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Calculating three-loop diagrams in heavy quark effective theory with integration-by-parts recurrence relations

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Abstract: An algorithm for calculation of three-loop propagator diagrams in HQET, based on integration-by-parts recurrence relations, is constructed and implemented as a REDUCE package Grinder, and in Axiom.

Keywords: Renormalization Regularization and Renormalons, Heavy Quarks Physics, QCD, NLO Computations.

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Contents

1. Introduction

Perturbative quantum field theory is progressing fast. New high-precision experiments require calculation of higher radiative corrections — multiloop Feynman diagrams. Recently, some calculations have been done which would seem impossible only a few years ago. This is due to the high degree of automation of the process of generation, analyses and calculation of Feynman diagrams, which is achieved via extensive use of computer algebra (see [1] for review and references). All such calculations are performed in the framework of dimensional regularization [2], i.e. diagrams are calculated as analytical functions of the space-time dimension $d = 4 - 2\epsilon$.

The integration-by-parts method [3] was invented for calculation of three-loop massless propagator diagrams. It is the most systematic method of those currently used, and the most appropriate for computer-algebra implementation. It was first implemented as a SCHOONSCHIP [4] package MINCER [5], and later re-implemented [6] in FORM [7] (in fact, FORM was created mainly to run MINCER). Since then, MINCER has been the engine behind most of spectacular successes of perturbative field theory. Some of these calculations, with gigabyte-size intermediate expressions, are among the largest computer-algebra calculations ever undertaken.

Integration-by-parts was used for other classes of problems, too. Many interesting physical results have been obtained with packages for calculating two-loop

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on-shell massive diagrams and three-loop vacuum diagrams with a single mass (see, e.g., [8, 9, 10]). Reduction of two-loop propagator integrals with generic masses and momentum to a finite set of bases integrals has been achieved [12]. First three-loop on-shell calculations have been done recently [11, 13].

Several years ago, an interesting new approach to heavy-quark problems in Quantum Chromodynamics has been formulated — Heavy Quark Effective Theory (HQET), see, e.g., [14, 15] for review and references. In collaboration with David Broadhurst, I applied the integration-by-parts method for calculating two-loop propagator diagrams in HQET [16]. Since then, the algorithm suggested was used in a large number of physics applications. A short review of the integration-by-parts method as applied in heavy quark physics is presented in [17].

In the present work, I apply this method for calculating three-loop propagator diagrams in HQET. Three-loop anomalous dimensions and spectral densities in HQET are necessary for a number of physics applications, such as improved extraction of the B meson decay constant f_B from lattice simulations and from QCD sum rules. HQET lagrangean does not involve mass in the leading order, and, therefore, the problem is quite similar to the massless one. My aim is to produce a reliable package for three-loop HQET calculations, which could play the same role as MINCER in massless theories. I call it Grinder. Some complication comes from the fact that there are two kinds of lines now — massless propagators and infinitely-heavy ones, and hence the number of diagram topologies is substantially larger.

In section 2, we consider three-loop HQET propagator diagrams with one- or two-loop massless or HQET propagator subdiagrams. To this end, we first recall well-known results for massless [3] and HQET [16] one- and two-loop diagrams. After that, diagrams with two-loop insertions, or with two one-loop insertions, are easily calculated. Two-loop HQET diagrams with a single one-loop insertion are dealt with in a manner similar to the plain two-loop diagrams. In section 3, we consider proper three-loop HQET propagator diagrams. Some details of implementation and testing are presented in section 4.

2. Diagrams with lower-loop propagator insertions

2.1 Two-loop massless propagator diagrams

The one-loop massless propagator integral (figure 1) can be easily calculated by using the Feynman parameterization or Fourier transform to the coordinate space and back:

$$\begin{aligned} \int \frac{d^d k}{D_1^{n_1} D_2^{n_2}} &= i\pi^{d/2} (-p^2)^{d/2-n_1-n_2} G(n_1, n_2)\,, \\ D_1 &= -k^2\,, \qquad D_2 = -(k+p)^2\,, \\ G(n_1, n_2) &= \frac{\Gamma(n_1+n_2-d/2)\Gamma(d/2-n_1)\Gamma(d/2-n_2)}{\Gamma(n_1)\Gamma(n_2)\Gamma(d-n_1-n_2)}\,. \end{aligned} \tag{2.1}$$

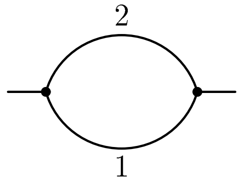


Figure 1: One-loop massless propagator diagram.

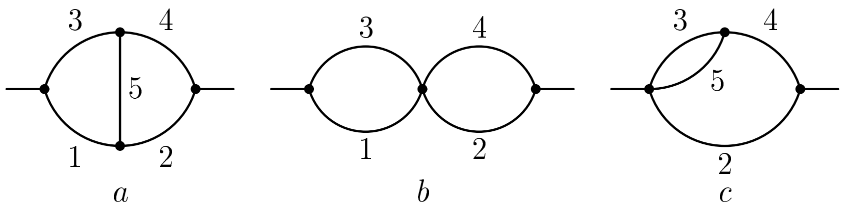


Figure 2: Two-loop massless propagator diagram

The integral with a numerator can be written as a finite sum [3]

$$\int \frac{P_n(k) d^d k}{D_1^{n_1} D_2^{n_2}} = i\pi^{d/2} (-p^2)^{d/2-n_1-n_2} \times$$
$$\times \sum_m G(n_1, n_2; n, m) \frac{(-p^2)^m}{m!} \left(-\frac{1}{4} \frac{\partial}{\partial k_\mu} \frac{\partial}{\partial k^\mu} \right)^m P_n(k) \bigg|_{k \to p}, \quad (2.2)$$
$$G(n_1, n_2; n, m) = \frac{\Gamma(n_1 + n_2 - m - d/2)\Gamma(d/2 - n_1 + n - m)\Gamma(d/2 - n_2 + m)}{\Gamma(n_1)\Gamma(n_2)\Gamma(d - n_1 - n_2 + n)},$$

where $P_n(k)$ is an arbitrary homogeneous polynomial: $P_n(\lambda k) = \lambda^n P_n(k)$.

We write the two-loop propagator integral (figure 2a) as

$$\int \frac{d^d k_1 d^d k_2}{D_1^{n_1} D_2^{n_2} D_3^{n_3} D_4^{n_4} D_5^{n_5}} = -\pi^d (-p^2)^{d - \sum n_i} G(n_1, n_2, n_3, n_4, n_5),$$
$$D_1 = -k_1^2, \quad D_2 = -k_2^2, \quad D_3 = -(k_1 + p)^2, \quad D_4 = -(k_2 + p)^2,$$
$$D_5 = -(k_1 - k_2)^2. \quad (2.3)$$

It is symmetric with respect to $1 \leftrightarrow 2$, $3 \leftrightarrow 4$, and also $1 \leftrightarrow 3$, $2 \leftrightarrow 4$. If one of the indices is zero, it can be easily calculated using (2.1) (figure 2b,c)

$$G(n_1, n_2, n_3, n_4, 0) = G(n_1, n_3) G(n_2, n_4), \quad (2.4)$$
$$G(0, n_2, n_3, n_4, n_5) = G(n_3, n_5) G(n_2, n_4 + n_3 + n_5 - d/2) \quad (2.5)$$

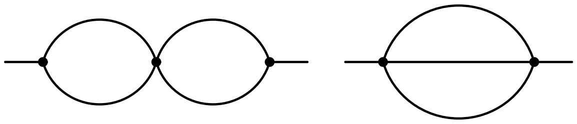
(and symmetric relations).

Applying the operators $\partial_1 \cdot (k_1 - k_2)$ and $\partial_1 \cdot k_1$ (where $\partial_i = \partial/\partial k_i$) to the integrand of (2.3), we obtain the recurrence relations for $G(n_1, n_2, n_3, n_4, n_5)$ (known as triangle relations [3])

$$\left[d - n_1 - n_3 - 2n_5 + n_1 \mathbf{1}^+ (\mathbf{2}^- - \mathbf{5}^-) + n_3 \mathbf{3}^+ (\mathbf{4}^- - \mathbf{5}^-) \right] G = 0, \quad (2.6)$$
$$\left[d - n_3 - n_5 - 2n_1 + n_3 \mathbf{3}^+ (1 - \mathbf{1}^-) + n_5 \mathbf{5}^+ (\mathbf{2}^- - \mathbf{1}^-) \right] G = 0, \quad (2.7)$$

where, for example,

$$\mathbf{1}^\pm G(n_1, n_2, n_3, n_4, n_5) = G(n_1 \pm 1, n_2, n_3, n_4, n_5). \quad (2.8)$$



a b

Figure 3: Basis two-loop massless propagator integrals

Of course, more relations are obtained by symmetry. Another interesting relation is derived by applying the operator $\frac{\partial}{\partial p} \cdot (k_2 + p)$. Substituting the general form of the relevant vector integral, we arrive at

$$\left[\frac{1}{2}d + n_4 - n_1 - n_2 - n_5 + \left(\frac{3}{2}d - n_1 - n_2 - n_3 - n_4 - n_5\right)(\mathbf{4}^- - \mathbf{2}^-) + \right. \qquad (2.9)$$
$$\left. + n_3\mathbf{3}^+(\mathbf{4}^- - \mathbf{5}^-)\right] G = 0\,.$$

This formula was derived long ago by S.A. Larin in his M.Sc. thesis [18] (again, similar relations follow by symmetries).

If indices of two adjacent lines are non-positive integers, the integral contains a no-scale vacuum subdiagram and hence vanishes. The cases with zero indices are given by (2.4), (2.5). When $n_5 < 0$ and $n_3 \neq 1$, n_5 can be raised by (2.9); the cases $n_1 \neq 1$, $n_2 \neq 1$, $n_4 \neq 1$ are symmetric. The case $n_5 < 0$, $n_1 = n_1 = n_3 = n_4 = 1$ is handled by

$$\begin{aligned}\left[(d - 2n_5 - 4)\mathbf{5}^+ + 2(d - n_5 - 3)\right] G(1,1,1,1,n_5) &= \\ = 2\mathbf{1}^+(\mathbf{3}^- - \mathbf{2}^-\mathbf{5}^+)G(1,1,1,1,n_5)\,, &\qquad (2.10)\end{aligned}$$

which follows from (2.6) and (2.7) at $n_1 = n_2 = n_3 = n_4 = 1$ (note that the terms in the right-hand side of (2.10) are trivial for any n_5; for $n_5 < 0$, they vanish). When $n_2 < 0$, it can be raised by (2.9); the cases $n_1 < 0$, $n_3 < 0$, $n_4 < 0$ are symmetric.

We are left with the most important situation when all the indices are positive. Applying (2.6), we reduce n_2, n_4, n_5 until one of them vanishes. Then (2.4) and (2.5) apply. If $\max(n_1, n_3) < \max(n_2, n_4)$, it is more efficient to lower n_1, n_3, n_5.

All one-loop integrals (figure 1) with integer $n_{1,2}$ are proportional to $G_1 = G(1,1)$, the coefficient being a rational function of d. All two-loop integrals with integer indices reduce to G_1^2 (figure 3a) and $G_2 = G(0,1,1,0,1)$ (figure 3b), with rational coefficients. Here

$$G_n = \frac{1}{\left(n + 1 - n\frac{d}{2}\right)_n \left((n+1)\frac{d}{2} - 2n - 1\right)_n} \frac{\Gamma(1 + n\epsilon)\Gamma^{n+1}(1-\epsilon)}{\Gamma(1 - (n+1)\epsilon)}\,, \qquad (2.11)$$

where $(x)_n = \Gamma(x+n)/\Gamma(x)$ is the Pochhammer symbol.

2.2 Two-loop HQET propagator diagrams

The one-loop HQET propagator integral (figure 4) can be easily calculated by using the modified Feynman parameterization (see, e.g., [16, 17]), or Fourier transform to the coordinate space and back:

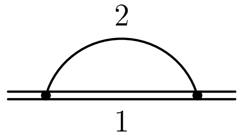


Figure 4: One-loop HQET propagator diagram.

$$\int \frac{d^d k}{D_1^{n_1} D_2^{n_2}} = i\pi^{d/2}(-2\omega)^{d-2n_2} I(n_1,n_2)\,,$$
$$D_1 = \frac{(k+p)\cdot v}{\omega}\,, \qquad D_2 = -k^2\,,$$
$$I(n_1,n_2) = \frac{\Gamma(n_1+2n_2-d)\Gamma(d/2-n_2)}{\Gamma(n_1)\Gamma(n_2)} \tag{2.12}$$

(here v is 4-velocity of the heavy quark, $v^2=1$, and $\omega = p\cdot v$ is the residual energy). Similarly to (2.2), we obtain

$$\int \frac{P_n(k) d^d k}{D_1^{n_1} D_2^{n_2}} = i\pi^{d/2}(-2\omega)^{d-2n_2} \times$$
$$\times \sum_m I(n_1,n_2;n,m) \frac{(-2\omega)^{2m}}{m!} \left(-\frac{1}{4}\frac{\partial}{\partial k_\mu}\frac{\partial}{\partial k^\mu}\right)^m P_n(k)\bigg|_{k\to 2\omega v}\,,$$
$$I(n_1,n_2;n,m) = \frac{\Gamma(n_1+2n_2-n-d)\Gamma(d/2-n_2+n-m)}{\Gamma(n_1)\Gamma(n_2)}\,. \tag{2.13}$$

There are two topologies of two-loop propagator HQET diagrams (figure 5a,b). We write the first of them as

$$\int \frac{d^d k_1 d^d k_2}{D_1^{n_1} D_2^{n_2} D_3^{n_3} D_4^{n_4} D_5^{n_5}} = -\pi^d(-2\omega)^{2(d-n_3-n_4-n_5)} I(n_1,n_2,n_3,n_4,n_5)\,,$$
$$D_1 = \frac{(k_1+p)\cdot v}{\omega}\,, \qquad D_2 = \frac{(k_2+p)\cdot v}{\omega}\,,$$
$$D_3 = -k_1^2\,, \qquad D_4 = -k_2^2\,, \qquad D_5 = -(k_1-k_2)^2\,. \tag{2.14}$$

It is symmetric with respect to $1 \leftrightarrow 3$, $2 \leftrightarrow 4$. If one of the indices is zero, it can be easily calculated using (2.12) and (2.1) (figure 5c,d,e)

$$I(n_1,n_2,n_3,n_4,0) = I(n_1,n_3) I(n_2,n_4)\,, \tag{2.15}$$
$$I(0,n_2,n_3,n_4,n_5) = G(n_3,n_5) I(n_2, n_4+n_3+n_5-d/2)\,, \tag{2.16}$$
$$I(n_1,n_2,0,n_4,n_5) = I(n_1,n_5) I(n_2+n_1+2n_5-d, n_4) \tag{2.17}$$

(and symmetric relations).

Applying the operators $\partial_1\cdot(k_1-k_2)$, $\partial_1\cdot k_1$ and $\partial_1\cdot v$ to the integrand of (2.14), we obtain [16]

$$\left[d-n_1-n_3-2n_5+n_1\mathbf{1}^+\mathbf{2}^-+n_3\mathbf{3}^+(\mathbf{4}^- - \mathbf{5}^-)\right] I = 0\,, \tag{2.18}$$
$$\left[d-n_1-n_5-2n_3+n_1\mathbf{1}^+ + n_5\mathbf{5}^+(\mathbf{4}^- - \mathbf{3}^-)\right] I = 0\,, \tag{2.19}$$
$$\left[-2n_1\mathbf{1}^+ + n_3\mathbf{3}^+(\mathbf{1}^- - 1) + n_5\mathbf{5}^+(\mathbf{1}^- - \mathbf{2}^-)\right] I = 0\,. \tag{2.20}$$

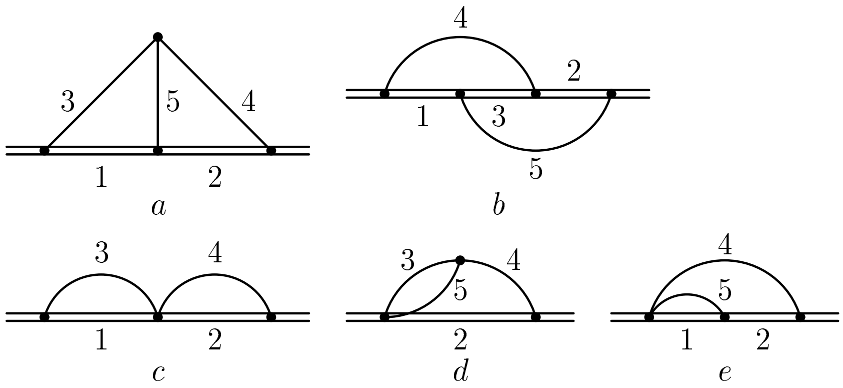


Figure 5: Two-loop HQET propagator diagram

Of course, more relations are obtained by symmetry. Applying $\omega\frac{d}{d\omega}$ and using homogeneity in ω, we obtain

$$\left[2(d-n_3-n_4-n_5)-n_1-n_2+n_1\mathbf{1}^+ +n_2\mathbf{2}^+\right] I=0\,, \tag{2.21}$$

which is nothing but the sum of (2.19) and its mirror-symmetric. Subtracting the $\mathbf{2}^-$ shifted version of (2.21) from (2.18), we obtain the most useful relation [16]

$$\begin{aligned}&\Big[d-n_1-n_2-n_3-2n_5+1-\\ &-(2(d-n_3-n_4-n_5)-n_1-n_2+1)\mathbf{2}^- +n_3\mathbf{3}^+(\mathbf{4}^- -\mathbf{5}^-)\Big] I=0\,,\end{aligned} \tag{2.22}$$

which lowers n_2, n_4, n_5, and does not raise heavy-quark indices.

If indices of two adjacent lines are non-positive integers, the integral contains a no-scale vacuum subdiagram and hence vanishes. The cases with zero indices are given by (2.15), (2.16) and (2.17). When $n_2<0$, it can be raised by (2.22); the case $n_1<0$ is symmetric. Similarly, if $n_3<0$, it can be raised by (2.22); the case $n_4<0$ is symmetric. When $n_5<0$ and $n_3\neq 1$, n_5 can be raised by (2.22) (the case $n_4\neq 1$ is symmetric); when $n_5<0$ and $n_1\neq 1$, n_5 can be raised by (2.19) (the case $n_2\neq 1$ is symmetric). The case $n_5<0$, $n_1=n_1=n_3=n_4=1$ is handled by

$$\begin{aligned}&\left[(d-2n_5-4)\mathbf{5}^+ -2(d-n_5-3)\right] I(1,1,1,1,n_5)=\\ &\quad=\left[(2d-2n_5-7)\mathbf{1}^-\mathbf{5}^+ -\mathbf{3}^-\mathbf{4}^+\mathbf{5}^- +\mathbf{1}^-\mathbf{3}^+\right] I(1,1,1,1,n_5)\,,\end{aligned} \tag{2.23}$$

which follows from (2.18), (2.19) and (2.20) at $n_1=n_2=n_3=n_4=1$ (note that the terms on the right-hand side of (2.23) are trivial for any n_5; for $n_5<0$, they vanish).

We are left with the most important situation when all the indices are positive. Applying (2.22), we reduce n_2, n_4, n_5 until one of them vanishes. Then (2.15), (2.16) and (2.17) apply. If $\max(n_1,n_3)<\max(n_2,n_4)$, it is more efficient to lower n_1, n_3, n_5.

In the second topology (figure 5b), three heavy-quark denominators depend on only two variables $k_{1,2}\cdot v$, hence they are linearly dependent. Therefore, there is one scalar product which cannot be expressed via the denominators. Let's define the integral

$$\int \frac{N^{n_0} d^d k_1 d^d k_2}{D_1^{n_1} D_2^{n_2} D_3^{n_3} D_4^{n_4} D_5^{n_5}} = -\pi^d (-2\omega)^{2(d+n_0-n_4-n_5)} J(n_1, n_2, n_3, n_4, n_5; n_0)\,,$$
$$D_1 = \frac{(k_1+p)\cdot v}{\omega}\,, \qquad D_2 = \frac{(k_2+p)\cdot v}{\omega}\,, \qquad D_3 = \frac{(k_1+k_2+p)\cdot v}{\omega}\,,$$
$$D_4 = -k_1^2\,, \qquad D_5 = -k_2^2\,, \qquad N = 2k_1\cdot k_2 \tag{2.24}$$

(it is symmetric with respect to $1 \leftrightarrow 2$, $4 \leftrightarrow 5$). Noting that $D_1 + D_2 - D_3 = 1$, we immediately have [16]

$$(1 - \mathbf{1}^- - \mathbf{2}^- + \mathbf{3}^-) J = 0\,. \tag{2.25}$$

Applying $\partial_1 \cdot k_1$, $\partial_1 \cdot k_2$ and $\partial_1 \cdot v$ to the integrand, we have

$$\left[d + n_0 - n_1 - n_3 - 2n_4 + n_1 \mathbf{1}^+ + n_3 \mathbf{3}^+ \mathbf{2}^-\right] J = 0 \tag{2.26}$$
$$\left[n_1 - n_3 - n_1 \mathbf{1}^+ \mathbf{3}^- + n_3 \mathbf{3}^+ \mathbf{1}^- + n_4 \mathbf{4}^+ \mathbf{0}^+ - 2n_0 \mathbf{0}^- \mathbf{5}^-\right] J = 0 \tag{2.27}$$
$$\left[-2n_1 \mathbf{1}^+ - 2n_3 \mathbf{3}^+ + n_4 \mathbf{4}^+ (\mathbf{1}^- - 1) + n_0 \mathbf{0}^- (\mathbf{2}^- - 1)\right] J = 0\,. \tag{2.28}$$

Homogeneity in ω gives $[2(d+n_0-n_4-n_5) - n_1 - n_2 - n_3 + n_1 \mathbf{1}^+ + n_2 \mathbf{2}^+ + n_3 \mathbf{3}^+] J = 0$, which is nothing but the sum of (2.26) and its mirror-symmetric. The boundary values of the integral (figure 5c,e) are

$$J(n_1, n_2, 0, n_4, n_5; n_0) = (\mathbf{5}^- - \mathbf{3}^- - \mathbf{4}^-)^{n_0} I(n_1, n_2, n_4, n_5, 0) \tag{2.29}$$
$$J(0, n_2, n_3, n_4, n_5; n_0) = (\mathbf{4}^- - \mathbf{3}^- + \mathbf{5}^-)^{n_0} I(n_3, n_2, 0, n_5, n_4) \tag{2.30}$$

and the symmetric relation for $n_2 = 0$. If $n_0 = 0$,

$$J(n_1, n_2, 0, n_4, n_5) = I(n_1, n_4) I(n_2, n_5) \tag{2.31}$$
$$J(0, n_2, n_3, n_4, n_5) = I(n_3, n_4) I(n_2 + n_3 + 2n_4 - d, n_5)\,. \tag{2.32}$$

If $n_4 \le 0$, or $n_5 \le 0$, or two adjacent heavy-quark indices are non-positive, the integral vanishes. If any of n_1, n_2, n_3 is negative, it can be raised by (2.25). If all of them are positive, we use (2.25) to lower n_1, n_2 or n_3, until one of these indices vanish.

Instead of using (2.29), (2.30) when $n_0 > 0$, we could proceed in another way. If $n_4 > 1$, we lower both n_0 and n_4 using (2.27) (the case $n_5 > 1$ is symmetric). We are left with $J(n_1, n_2, 0, 1, 1; n_0)$ and $J(0, n_2, n_3, 1, 1; n_0)$. In the first case, if $n_1 > 1$, we lower it using (2.28) (the case $n_2 > 1$ is symmetric). In the second case, if $n_3 > 1$, we lower it using (2.28); if $n_2 > 1$, we lower it using the difference of (2.28) and its mirror-symmetric. The two remaining cases (figure 5c,e) are easily evaluated using (2.13):

$$J(1, 1, 0, 1, 1; n_0) = (-1)^{n_0} I(1, 1; n_0, 0) I(1, 1)\,,$$
$$J(0, 1, 1, 1, 1; n_0) = I(1, 1; n_0, 0) \sum_{l=0}^{n_0} \frac{(-1)^l n_0!}{l!(n_0 - l)!} I(4 - 2n_0 + l - d, 1)$$

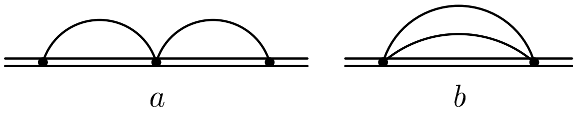


Figure 6: Basis two-loop HQET propagator integrals

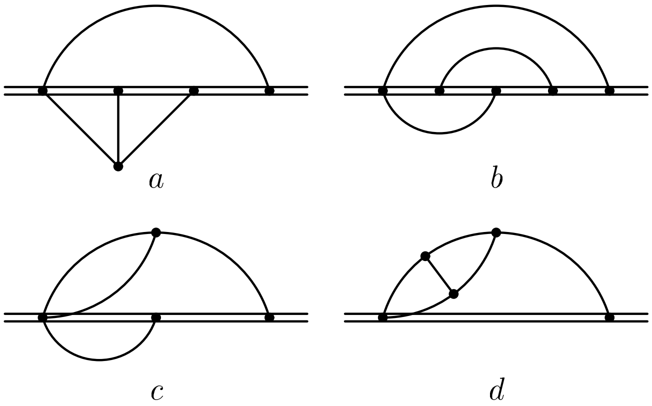


Figure 7: Three-loop HQET propagator diagrams with a two-loop propagator insertion, or with two one-loop insertions

All one-loop integrals (figure 4) with integer $n_{1,2}$ are proportional to $I_1 = I(1,1)$, the coefficient being a rational function of d. All two-loop integrals with integer indices reduce to I_1^2 (figure 6a) and $I_2 = I(0,1,1,0,1)$ (figure 6b), with rational coefficients. Here

$$I_n = \frac{1}{(1-n(d-2))_{2n}} \Gamma(1+2n\epsilon)\Gamma^n(1-\epsilon)\,. \tag{2.33}$$

2.3 Three-loop HQET diagrams with lower-loop insertions

Diagrams with a two-loop propagator insertion, or with two one-loop insertions, figure 7, are trivially calculated by multiplying the relevant insertion[s] by the one-loop HQET integral with non-integer indices, whose values are obvious by dimensionality.

Next we consider the diagrams obtained from the two-loop HQET diagram of figure 5a by adding a single one-loop propagator insertion (figure 8). They are equal to the product of the corresponding one-loop integral and the integral of figure 5a with one non-integer index. All relations derived for the diagram of figure 5a are valid; however, the non-integer index changes the strategy of their application.

Let's consider the case of figure 8a with a non-integer n_3. When $n_5 < 0$, it can always be raised by (2.22). When $n_2 < 0$, it can be raised by (2.22), too; when $n_1 < 0$, it can be raised by the mirror-symmetric relation. When $n_4 < 0$, it can always be raised by (2.22), too. Finally, when all indices n_1, n_2, n_4, n_5 are positive, we use (2.22) to lower n_2, n_4 or n_5, until (2.15), (2.16) and (2.17) are reached.

In the case of figure 8b, n_1 is non-integer. When $n_5 < 0$ and $n_3 \neq 1$, we can raise n_5 by (2.22) (the case $n_4 \neq 1$ is symmetric); when $n_5 < 0$ and $n_2 \neq 1$, we can lower

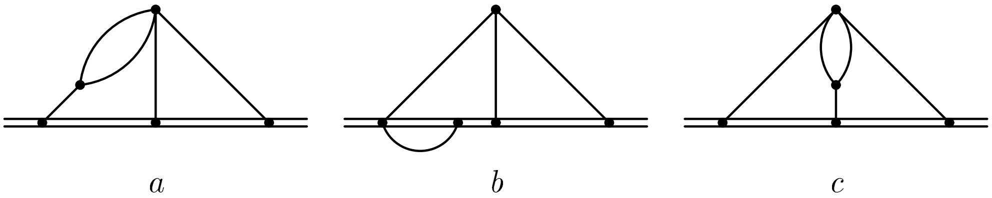


Figure 8: The diagram of figure 5a with a single one-loop insertion

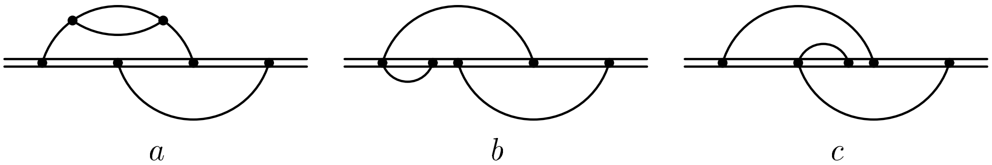


Figure 9: The diagram of figure 5b with a single one-loop insertion

n_2 by the relation symmetric to (2.19); when $n_5 < 0$ and $n_2 = n_3 = n_4 = 1$, we can use the relation

$$\left[2(d - n_5 - 3) - \mathbf{4}^+ - n_5\mathbf{5}^+\mathbf{1}^-\right] I(n_1, 1, 1, 1, n_5) = 0\,, \tag{2.34}$$

which follows from (2.20) and (2.19), and then the term with $\mathbf{4}^+$ can be treated as above. When $n_2 < 0$, it can be raised by (2.22). When $n_3 < 0$ and $n_4 \neq 1$, we can raise n_3 by the relation symmetric to (2.22); when $n_3 < 0$ and $n_5 \neq 1$, we can raise n_3 by (2.19); when $n_3 < 0$ and $n_2 \neq 1$, we lower n_2 by (2.21); finally, when $n_3 < 0$ and $n_2 = n_4 = n_5 = 1$, we use (2.22) to get a trivial term with $n_2 = 0$. When $n_4 < 0$ and $n_3 \neq 1$, we can raise n_4 by (2.22); when $n_4 < 0$ and $n_5 \neq 1$, we can raise n_4 by (2.19); when $n_4 < 0$ and $n_2 \neq 1$, we lower n_2 by (2.21); finally, when $n_4 < 0$ and $n_2 = n_3 = n_5 = 1$, we use (2.20) to raise n_3 or n_5, and proceed as above. Finally, when all indices n_2, n_3, n_4, n_5 are positive, we use (2.22) to lower n_2, n_4 or n_5, until (2.15), (2.16) and (2.17) are reached.

In the case of figure 8c, n_5 is non-integer. When $n_2 < 0$, it can be raised by (2.22); the case $n_1 < 0$ is symmetric. When $n_3 < 0$, it can be raised by (2.19); the case $n_4 < 0$ is symmetric. When $n_1 > 1$, it can be lowered by (2.19); the case $n_2 > 1$ is symmetric. When $n_3 > 1$, it can be lowered by (2.20), with a trivial additional term having $n_1 = 0$; the case $n_4 > 1$ is symmetric. We are left with $n_1 = n_2 = n_3 = n_4 = 1$; the relation (2.23) can be used to lower or raise n_5, with trivial additional terms. An integral with some specific value of n_5 (of the form integer plus ϵ) has to be considered as a new basis element. This integral has been calculated, exactly in d dimensions, in [19] in terms of ${}_3F_2$ hypergeometric functions, using the Hegenbauer polynomial technique in the coordinate space [20].

Finally, we consider the diagrams obtained from the two-loop HQET diagram of figure 5b by adding a single one-loop propagator insertion (figure 9). The diagram of figure 9a, with a non-integer n_4 is calculated exactly as the two-loop one.

In the case of figure 9b, n_1 is non-integer. When $n_3 < 0$ or $n_2 < 0$, they can be raised by (2.25). In the case of positive indices, we use

$$\left[d - n_3 - 2n_4 + n_1 \mathbf{1}^+(\mathbf{2}^- - \mathbf{3}^-) + n_3 \mathbf{3}^+\mathbf{2}^-\right] J = 0 \tag{2.35}$$

(which follows from (2.26) and (2.25)). It either lowers n_2+n_3, or, at a fixed n_2+n_3, lowers n_2. Therefore, sooner or later, we reach (2.29) and (2.30).

In the case of figure 9c, n_3 is non-integer. If $n_1 < 0$ or $n_2 < 0$, they can be raised by (2.25). When $n_4 > 1$, we can lower it or n_1 by (2.28); the case $n_5 > 1$ is symmetric. When $n_1 > 1$, it can be lowered by (2.26); the case $n_2 > 1$ is symmetric. We are left with $n_1 = n_2 = n_4 = n_5 = 1$; the relation (2.25) can be used to lower or raise n_3, with trivial additional terms. An integral with some specific value of n_3 (of the form integer plus 2ϵ) has to be considered as a new basis element.

It is not difficult to calculate $J(1,1,n_3,n_4,n_5)$ for arbitrary $n_{3,4,5}$ (not necessarily integer) in the coordinate space:

$$\begin{aligned}
J(1,1,n,n_1,n_2) &= \frac{\Gamma(n-2(d-n_1-n_2-1))\Gamma(d/2-n_1)\Gamma(d/2-n_2)}{\Gamma(n)\Gamma(n_1)\Gamma(n_2)} J\,, \\
J &= t^{2(d-n_1-n_2)-n-1} \int\limits_{0<t_1<t_2<t} dt_1\, dt_2\, t_2^{2n_1-d}(t-t_1)^{2n_2-d}(t_2-t_1)^{n-1} \\
&= \frac{1}{n(2n_1+n+1-d)}\, {}_3F_2\left(\left. \begin{array}{c} 1,\ d-2n_2,\ 2n_1+n+1-d \\ n+1,\ 2n_1+n+2-d \end{array} \right| 1 \right) . \qquad (2.36)
\end{aligned}$$

All diagrams considered in this section are particular cases of the generic three-loop topologies, which will be discussed in section 3, when some lines are shrunk (i.e., some indices vanish). Therefore, we don't consider diagrams with numerators here: numerators should be dealt with in the context of generic topologies, and the formulae of this section are used only as boundary values for the corresponding recurrence relations, after elimination of numerators.

All three-loop HQET propagator integrals with lower-loop propagator insertions are linear combinations of 7 basis integrals (figure 10a–g), coefficients being rational functions of d. The basis integrals of figure 10a (I_1^3), figure 10b ($I_1 I_2$), figure 10c (I_3), figure 10d ($I_3 I_1^2/I_2$), and figure 10e ($I_3 G_1^2/G_2$) are known exactly in d dimensions in terms of Γ functions. Those of figure 10f,g contain hypergeometric ${}_3F_2$ functions of the unit argument. Several terms of their expansion in ϵ can be obtained using the methods which were recently developed in [21]. As we shall see in section 3, there is only one additional basis integral, figure 10h.

3. Proper three-loop HQET propagator diagrams

In the massless case [3], there are only 3 topologies of proper three-loop propagator diagrams: Mercedez, Ladder, and Non-planar (plus their reduced forms obtained by shrinking some lines). Now we have 10 topologies instead (figure 11). Each of

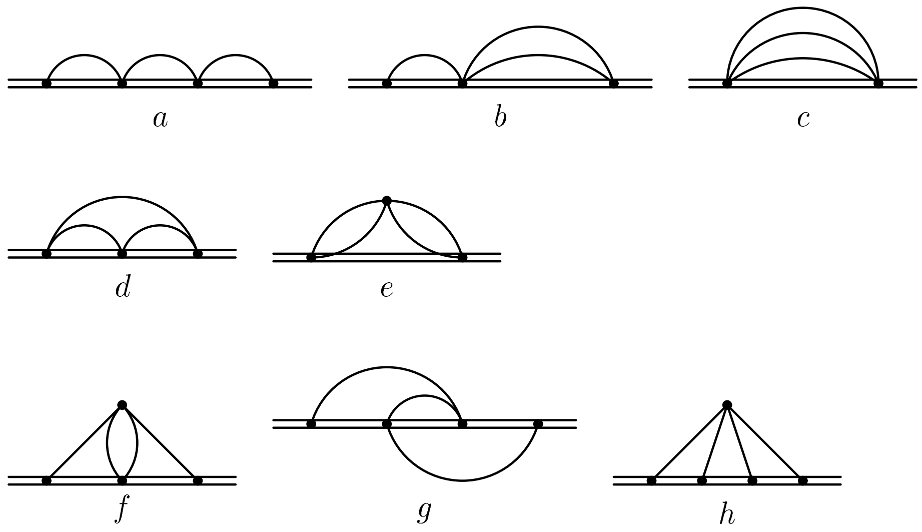


Figure 10: Basis tree-loop HQET propagator integrals

them has 8 propagators. In the diagrams with 4 heavy-quark lines (figure 11e–g), there is one linear dependence between their denominators; with 5 heavy-quark lines (figure 11h–j) — 2 dependences. There are 9 independent scalar products of 3 loop momenta $k_{1,2,3}$ and the 4-velocity v. Therefore, in the diagrams with two or three heavy-quark lines (figure 11a–d), there is one scalar product in the numerator which cannot be cancelled against the denominators; with 4 heavy-quark lines (figure 11e–g) — two scalar products; with 5 heavy-quark lines (figure 11h–j) — three scalar products.

When calculating these diagrams using recurrence relations, some indices may vanish. This corresponds to shrinking the corresponding lines. In some cases, this results in diagrams with lower-loop propagator insertions, which were calculated in section 2. The diagrams of figure 12 are still non-trivial.

3.1 Diagram with two heavy-quark lines

Let's consider the diagram of figure 11a first. We define

$$\begin{aligned}
&\int \frac{N^{n_0} d^d k_1 d^d k_2 d^d k_3}{D_1^{n_1} D_2^{n_2} D_3^{n_3} D_4^{n_4} D_5^{n_5} D_6^{n_6} D_7^{n_7} D_8^{n_8}} = \\
&\quad = -i\pi^{3d/2} (-2\omega)^{3d+2n_0-2\sum_{i=3}^{8} n_i} I_a(n_1, n_2, n_3, n_4, n_5, n_6, n_7, n_8; n_0)\,, \\
&N = \frac{k_3 \cdot v}{\omega}\,, \qquad D_1 = \frac{(k_1+p)\cdot v}{\omega}\,, \qquad D_2 = \frac{(k_2+p)\cdot v}{\omega}\,, \\
&D_3 = -k_1^2\,, \qquad D_4 = -k_2^2\,, \qquad D_5 = -(k_1-k_2)^2\,, \\
&D_6 = -k_3^3\,, \qquad D_7 = -(k_3+k_1)^2\,, \qquad D_8 = -(k_3+k_2)^2\,.
\end{aligned} \tag{3.1}$$

This integral is mirror-symmetric with respect to $1 \leftrightarrow 2$, $3 \leftrightarrow 4$, $7 \leftrightarrow 8$. It vanishes when the indices of the following groups of lines are non-positive: 12, 67, 68, 78, 375, 485, 315, 425, 364, 137, 248, 157, or 258.

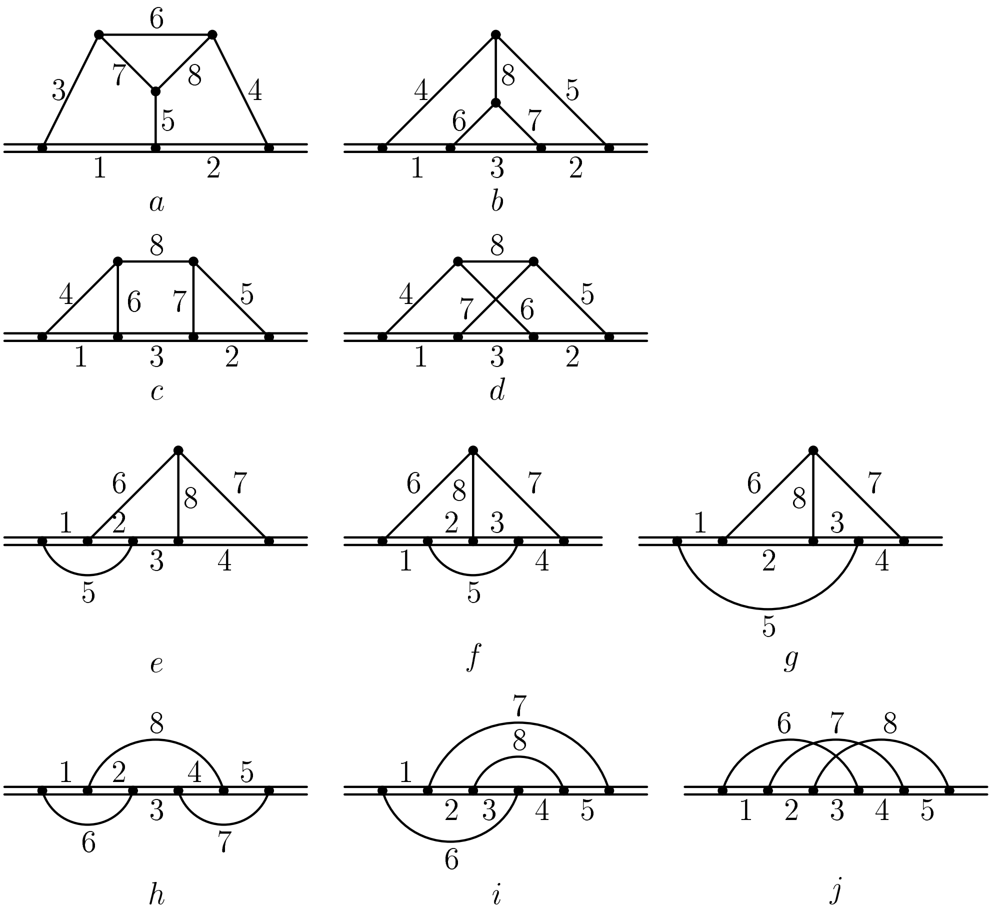


Figure 11: Topologies of proper three-loop HQET propagator diagrams: Mercedez (*a, b, f, g, i*), Ladder (*c, e, h*), Non-planar (*d, j*)

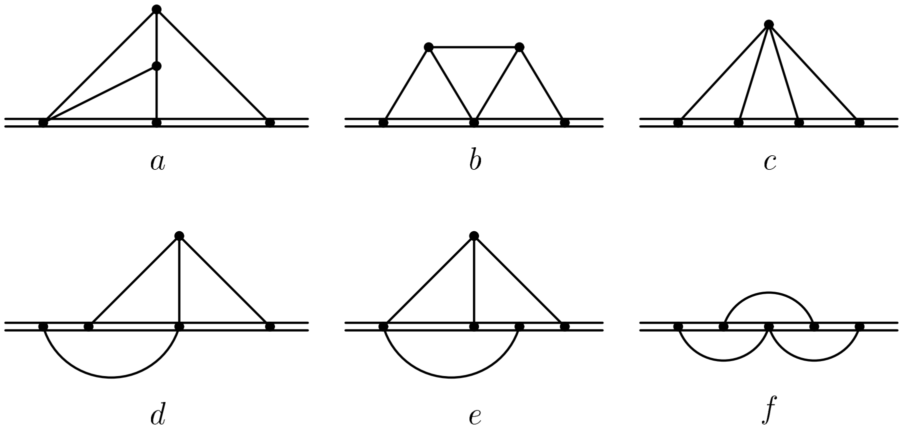


Figure 12: Topologies of non-trivial three-loop diagrams with a shrunk line

First we are going to get rid of the numerator. When $n_0 > 0$ and $n_7 \neq 1$, we can lower n_0 by

$$\left[-2n_1\mathbf{1}^+ + n_3\mathbf{3}^+(\mathbf{1}^- - 1) + n_5\mathbf{5}^+(\mathbf{1}^- - \mathbf{2}^-) + n_7\mathbf{7}^+(\mathbf{1}^- - 1 + \mathbf{0}^+)\right] I_a = 0\,, \quad (3.2)$$

which is obtained by applying $\partial_1 \cdot v$ to the integrand of (3.1); the case $n_0 > 0$, $n_8 \neq 1$ is symmetric. When $n_0 > 0$ and $n_6 \neq 1$, we can lower n_0 by

$$\Big[2\left[3d - n_1 - n_2 - 2(n_3 + n_4 + n_5 + n_6 + n_7 + n_8 - n_0)\right] + \\ + n_3\mathbf{3}^+(\mathbf{1}^- - 1) + n_4\mathbf{4}^+(\mathbf{2}^- - 1) - n_6\mathbf{6}^+\mathbf{0}^+ - 2n_0\mathbf{0}^-\Big] I_a = 0\,, \quad (3.3)$$

which is the $(\partial_1 + \partial_2 - \partial_3) \cdot v$ relation simplified using the homogeneity relation

$$\left[3d - n_1 - n_2 - 2(n_3 + n_4 + n_5 + n_6 + n_7 + n_8 - n_0) + n_1\mathbf{1}^+ + n_2\mathbf{2}^+\right] I_a = 0\,. \quad (3.4)$$

When $n_0 > 0$ and $n_1 \neq 1$, we can lower n_0 by

$$\Big[d - n_1 - n_3 - n_5 - 2n_7 + n_1\mathbf{1}^+(1 - \mathbf{0}^+) + \\ + n_3\mathbf{3}^+(\mathbf{6}^- - \mathbf{7}^-) + n_5\mathbf{5}^+(\mathbf{8}^- - \mathbf{7}^-)\Big] I_a = 0\,, \quad (3.5)$$

which is obtained by applying $\partial_1 \cdot (k_1 + k_3)$ to the integrand of (3.1); the case $n_0 > 0$, $n_2 \neq 1$ is symmetric. We are left with $n_0 > 0$, $n_1 = n_2 = n_6 = n_7 = n_8 = 1$; we use $\partial_3 \cdot (k_3 + k_1)$ relation

$$\Big[d + n_0 - n_6 - n_8 - 2n_7 + n_0\mathbf{0}^-(\mathbf{1}^- - 1) + \\ + n_6\mathbf{6}^+(\mathbf{3}^- - \mathbf{7}^-) + n_8\mathbf{8}^+(\mathbf{5}^- - \mathbf{7}^-)\Big] I_a = 0 \quad (3.6)$$

to lower n_0 or raise n_6 or n_8, and apply the method described above again.

Now we shall discuss the integral $I_a(n_1, n_2, n_3, n_4, n_5, n_6, n_7, n_8)$ without numerator ($n_0 = 0$). Applying $\partial_3 \cdot k_3$, $\partial_1 \cdot k_1$, $\partial_1 \cdot (k_1 - k_2)$, $(\partial_1 + \partial_2 - \partial_3) \cdot k_1$ to the integrand of (3.1), we obtain the recurrence relations

$$\left[d - n_7 - n_8 - 2n_6 + n_7\mathbf{7}^+(\mathbf{3}^- - \mathbf{6}^-) + n_8\mathbf{8}^+(\mathbf{4}^- - \mathbf{6}^-)\right] I_a = 0\,, \quad (3.7)$$

$$\Big[d - n_1 - n_5 - n_7 - 2n_3 + n_1\mathbf{1}^+ + \\ + n_5\mathbf{5}^+(\mathbf{4}^- - \mathbf{3}^-) + n_7\mathbf{7}^+(\mathbf{6}^- - \mathbf{3}^-)\Big] I_a = 0\,, \quad (3.8)$$

$$\Big[d - n_1 - n_3 - n_7 - 2n_5 + n_1\mathbf{1}^+\mathbf{2}^- + \\ + n_3\mathbf{3}^+(\mathbf{4}^- - \mathbf{5}^-) + n_7\mathbf{7}^+(\mathbf{8}^- - \mathbf{5}^-)\Big] I_a = 0\,, \quad (3.9)$$

$$\Big[2(d - n_4 - n_5 - n_7 - n_8) - n_2 - n_6 + n_2\mathbf{2}^+\mathbf{1}^- + \\ + n_4\mathbf{4}^+(\mathbf{3}^- - \mathbf{5}^-) + n_6\mathbf{6}^+(\mathbf{3}^- - \mathbf{7}^-)\Big] I_a = 0\,, \quad (3.10)$$

where the last relation was simplified using (3.4).

The cases $n_1 = 0$, $n_2 = 0$, $n_6 = 0$, $n_7 = 0$, $n_8 = 0$ are trivial. When $n_1 < 0$, it can be raised by

$$[3d - n_1 - n_2 - 2(n_3 + n_4 + n_5 + n_6 + n_7 + n_8)]\, I_a = \tag{3.11}$$
$$= \left[d - n_1 - n_2 - n_4 - n_8 - 2n_5 + n_4\mathbf{4}^+(\mathbf{3}^- - \mathbf{5}^-) + n_8\mathbf{8}^+(\mathbf{7}^- - \mathbf{5}^-)\right] \mathbf{1}^+ I_a\,,$$

which is the difference of (3.4) and $\mathbf{1}^+$ shifted version of the relation symmetric to (3.9); the case $n_2 < 0$ is symmetric. When $n_6 < 0$ and $n_7 \neq 1$, we can raise n_6 by (3.8) (the case $n_8 \neq 1$ is symmetric); when $n_6 < 0$ and $n_7 = n_8 = 1$, we can raise n_6 or n_8 by (3.6). When $n_7 < 0$ and $n_8 \neq 1$, we can raise n_7 by the relation symmetric to (3.9); when $n_7 < 0$ and $n_6 \neq 1$, we can raise n_7 by (3.10); when $n_7 < 0$ and $n_6 = n_8 = 1$, we can raise n_7 or n_6 by the relation symmetric to (3.6). The case $n_8 < 0$ is symmetric. The case $n_3 = 0$ (figure 12a, $J_a(n_1, n_2, n_4, n_5, n_6, n_7, n_8)$) will be considered later in this section; the case $n_4 = 0$ is symmetric. When $n_3 < 0$ and $n_6 \neq 1$, we can raise n_3 by (3.6); when $n_3 < 0$ and $n_7 \neq 1$, we can raise n_3 by (3.7); when $n_3 < 0$ and $n_6 = n_7 = 1$, we can raise n_6 or n_7 by the relation symmetric to (3.6). The case $n_4 < 0$ is symmetric. The case $n_5 = 0$ (figure 12b, $J_b(n_1, n_2, n_3, n_4, n_6, n_7, n_8)$) will be considered later in this section. When $n_5 < 0$ and $n_8 \neq 1$, we can raise n_5 by (3.6) (the case $n_7 \neq 1$ is symmetric); when $n_5 < 0$ and $n_7 = n_8 = 1$, we can raise n_7 or n_8 by (3.7). When all the indices are positive, we can use (3.7) to kill one of the lines 3, 4, 6; or we can use (3.6) to kill one of the lines 3, 5, 7; or we can use the symmetric relation to kill one of the lines 4, 5, 8; or we can use (3.10) to kill one of the lines 1, 3, 5, 7; or we can use the symmetric relation to kill one of the lines 2, 4, 5, 8.

Now we consider $J_a(n_1, n_2, n_4, n_5, n_6, n_7, n_8) = I_a(n_1, n_2, 0, n_3, n_4, n_5, n_6, n_7, n_8)$ (figure 12a). This integral vanishes when the indices of the following groups of lines are non-positive: 12, 67, 68, 78, 57, 15, 17, 46, 485, 425, 248, or 258. If any index is zero, the integral becomes trivial. If $n_1 < 0$, $n_2 < 0$, $n_6 < 0$, $n_7 < 0$, or $n_8 < 0$, we just consider J_a as I_a with $n_3 = 0$ and proceed as usual; the integral reduces to trivial ones not including J_a. When $n_4 > 0$, we can kill one of the lines 2, 4, 8 using the relation symmetric to (3.10). When $n_5 > 0$, we can kill one of the lines 2, 5, 8 using (3.9). When both n_4 and n_5 are negative, we proceed as follows. If $n_6 \neq 1$, n_4 can be raised by the relation symmetric to (3.10); if $n_7 \neq 1$, n_5 can be raised by (3.9); n_2 can be lowered down to 1 using (3.4). The remaining integral can be calculated by a repeated use of (2.13):

$$\begin{aligned} &J_a(n_1, 1, -|n_4|, -|n_5|, 1, 1, n_8) = \\ &\qquad = \sum_{l_1,l_2,l_3,m} \frac{(-1)^{l_1+l_3}|n_4|!|n_5|!}{(|n_5| - l_1)!m!(l_1 - 2m)!(|n_4| - l_2)!l_3!(l_2 - l_3)!} \times \\ &\qquad\quad \times I(n_1, 1; l_1, m) I(1, n_8 - |n_4| - |n_5| + l_1 + l_2 - m; l_2, 0) \times \\ &\qquad\quad \times I(-2(d + |n_4| + |n_5| - n_8) + n_1 + l_3 + 3, 1)\,. \end{aligned}$$

Finally, we consider $J_b(n_1,n_2,n_3,n_4,n_6,n_7,n_8) = I_a(n_1,n_2,n_3,n_4,0,n_6,n_7,n_8)$ (figure 12b). This integral is mirror-symmetric; it vanishes when the indices of the following groups of lines are non-positive: 12, 67, 68, 78, 37, 48, 13, 17, 24, 28, or 364. If any index is zero, the integral becomes trivial. If $n_1<0$, $n_2<0$, $n_3<0$, $n_4<0$, $n_6<0$, $n_7<0$, or $n_8<0$, we just consider J_b as I_a with $n_5=0$ and proceed as usual; the integral reduces to trivial ones not including J_b (though, possibly, including J_a). When all the indices are positive, one of the lines 3, 4, 6 can be killed by (3.7).

3.2 Mercedez with three heavy-quark lines

Now let's consider the diagram of figure 11b. We define

$$\begin{gathered}
\int \frac{N^{n_0} d^d k_1 d^d k_2 d^d k_3}{D_1^{n_1} D_2^{n_2} D_3^{n_3} D_4^{n_4} D_5^{n_5} D_6^{n_6} D_7^{n_7} D_8^{n_8}} = \\
= -i\pi^{3d/2}(-2\omega)^{3d+2n_0-2\sum_{i=4}^{8} n_i} I_b(n_1,n_2,n_3,n_4,n_5,n_6,n_7,n_8;n_0)\,, \\
D_1 = \frac{(k_1+p)\cdot v}{\omega}\,, \qquad D_2 = \frac{(k_2+p)\cdot v}{\omega}\,, \qquad D_3 = \frac{(k_3+p)\cdot v}{\omega}\,, \\
D_4 = -k_1^2\,, \qquad D_5 = -k_2^2\,, \qquad D_6 = -(k_1-k_3)^2\,, \\
D_7 = -(k_2-k_3)^2\,, \qquad D_8 = -(k_1-k_2)^2\,, \qquad N = -k_3^2\,.
\end{gathered} \tag{3.12}$$

This integral is mirror-symmetric with respect to $1 \leftrightarrow 2$, $4 \leftrightarrow 5$, $6 \leftrightarrow 7$. It vanishes when the indices of the following groups of lines are non-positive: 36, 37, 67, 123, 148, 168, 416, 486, 258, 278, 527, or 587.

First we are going to get rid of the numerator. When $n_4 \neq 1$, we can lower n_0 by the $\partial_1 \cdot (k_1-k_3)$ relation

$$\left[d - n_1 - n_4 - n_8 - 2n_6 + n_1\mathbf{1}^+\mathbf{3}^- + n_4\mathbf{4}^+(\mathbf{0}^+ - \mathbf{6}^-) + n_8\mathbf{8}^+(\mathbf{7}^- - \mathbf{6}^-)\right] I_b = 0\,; \tag{3.13}$$

the case $n_5 \neq 0$ is symmetric. When $n_6 \neq 1$, we can lower n_0 by the $\partial_1 \cdot k_1$ relation

$$\left[d - n_1 - n_6 - n_8 - 2n_4 + n_1\mathbf{1}^+ + n_6\mathbf{6}^+(\mathbf{0}^+ - \mathbf{4}^-) + n_8\mathbf{8}^+(\mathbf{5}^- - \mathbf{4}^-)\right] I_b = 0\,; \tag{3.14}$$

the case $n_7 \neq 1$ is symmetric. When $n_3 \neq 1$, we can lower n_0 or raise n_6 or n_7 by the $\partial_3 \cdot v$ relation

$$\left[-2n_3\mathbf{3}^+ + n_0\mathbf{0}^-(1 - \mathbf{3}^-) + n_6\mathbf{6}^+(\mathbf{3}^- - \mathbf{1}^-) + n_7\mathbf{7}^+(\mathbf{3}^- - \mathbf{2}^-)\right] I_b = 0\,. \tag{3.15}$$

Finally, we can lower n_0 or raise n_3 or n_7 by the $\partial_3 \cdot (k_3-k_1)$ relation

$$\left[d + n_0 - n_3 - n_7 - 2n_6 + n_3\mathbf{3}^+\mathbf{1}^- + n_0\mathbf{0}^-(\mathbf{6}^- - \mathbf{4}^-) + n_7\mathbf{7}^+(\mathbf{8}^- - \mathbf{6}^-)\right] I_b = 0\,. \tag{3.16}$$

Now we shall discuss the integral $I_b(n_1,n_2,n_3,n_4,n_5,n_6,n_7,n_8)$ without numerator ($n_0=0$). Applying $\partial_1\cdot v$, $\partial_1\cdot(k_1-k_2)$, $(\partial_1+\partial_2+\partial_3)\cdot k_1$ to the integrand of (3.12), we obtain the recurrence relations

$$\left[-2n_1\mathbf{1}^+ + n_4\mathbf{4}^+(\mathbf{1}^- - 1) + n_6\mathbf{6}^+(\mathbf{1}^- - \mathbf{3}^-) + n_8\mathbf{8}^+(\mathbf{1}^- - \mathbf{2}^-)\right] I_b = 0\,, \quad (3.17)$$

$$\Big[d - n_1 - n_4 - n_6 - 2n_8 + n_1\mathbf{1}^+\mathbf{2}^- + \\ + n_4\mathbf{4}^+(\mathbf{5}^- - \mathbf{8}^-) + n_6\mathbf{6}^+(\mathbf{7}^- - \mathbf{8}^-)\Big] I_b = 0\,, \quad (3.18)$$

$$\Big[2(d - n_6 - n_7 - n_8) - n_2 - n_3 - n_5 + \\ + (n_2\mathbf{2}^+ + n_3\mathbf{3}^+)\mathbf{1}^- + n_5\mathbf{5}^+(\mathbf{4}^- - \mathbf{8}^-)\Big] I_b = 0\,, \quad (3.19)$$

where the last relation was simplified using the homogeneity relation

$$\Big[3d - n_1 - n_2 - n_3 - 2(n_4+n_5+n_6+n_7+n_8) + \\ + n_1\mathbf{1}^+ + n_2\mathbf{2}^+ + n_3\mathbf{3}^+\Big] I_b = 0\,. \quad (3.20)$$

Using it to simplify the sum of (3.17), its symmetric and (3.15), we get

$$\Big[2\left[3d - n_1 - n_2 - n_3 - 2(n_4+n_5+n_6+n_7+n_8)\right] + \\ + n_4\mathbf{4}^+(\mathbf{1}^- - 1) + n_5\mathbf{5}^+(\mathbf{2}^- - 1)\Big] I_b = 0\,. \quad (3.21)$$

The cases $n_3=0$, $n_4=0$, $n_5=0$, $n_6=0$, $n_7=0$ are trivial. When $n_3<0$ and $n_6\neq 1$, we can raise n_3 by (3.17) (the case $n_7\neq 1$ is symmetric); when $n_3<0$ and $n_6=n_7=1$, we can raise n_3 or n_7 by (3.16). When $n_4<0$ and $n_5\neq 1$, we can raise n_4 by the relation symmetric to (3.18); when $n_4<0$ and $n_5=1$, we can raise n_4 or n_5 by (3.21). The case $n_5<0$ is symmetric. When $n_6<0$ and $n_7\neq 1$, we can raise n_6 by (3.16); when $n_6<0$ and $n_3\neq 1$, $n_7=1$, we can raise n_6 or n_7 by (3.15); when $n_6<0$ and $n_3=n_7=1$, we can raise n_6 or n_3 by the relation symmetric to (3.16). The case $n_7<0$ is symmetric. The case $n_1=0$ is $J_a(n_3,n_2,n_5,n_7,n_4,n_6,n_8)$ (figure 12a, section 3.1), $n_2=0$ is symmetric. When $n_1<0$ and $n_8\neq 1$, we can raise n_1 by the relation symmetric to (3.17); when $n_1<0$ and $n_6\neq 1$, we can raise n_1 by (3.15); when $n_1<0$ and $n_4\neq 1$, we can raise n_1 by (3.21); when $n_1<0$ and $n_2\neq 1$, we can raise n_1 by the relation symmetric to (3.18); when $n_1<0$ and $n_3\neq 1$, we can raise n_1 by (3.16); when $n_1<0$ and $n_2=n_3=n_4=n_6=n_8=1$, we can raise n_1, n_2 or n_3 by (3.20). The case $n_2<0$ is symmetric. The case $n_8=0$ (figure 12c, $J_c(n_1,n_2,n_3,n_4,n_5,n_6,n_7)$) will be considered later in this section. When $n_8<0$ and $n_7\neq 1$, we can raise n_8 by (3.16) (the case $n_6\neq 1$ is symmetric); when $n_8<0$ and $n_5\neq 1$, we can raise

n_8 by (3.19) (the case $n_4 \neq 1$ is symmetric); when $n_8 < 0$ and $n_1 \neq 1$, we can raise n_8 by (3.17) (the case $n_2 \neq 1$ is symmetric); when $n_8 < 0$ and $n_3 \neq 1$, we can raise n_8 by (3.15); when $n_8 < 0$ and $n_1 = n_2 = n_3 = n_4 = n_5 = n_6 = n_7 = 1$, we can raise n_1, n_2 or n_3 by (3.20). When all the indices are positive, we can kill one of the lines 1, 6, 8 by (3.16), or one of the lines 2, 7, 8 by its mirror-symmetric relation.

Now we consider $J_c(n_1, n_2, n_3, n_4, n_5, n_6, n_7) = I_b(n_1, n_2, n_3, n_4, n_5, n_6, n_7, n_8)$ (figure 12c). It is mirror-symmetric; it vanishes if any two indices of n_4, n_5, n_6, n_7 are non-positive, or n_1, n_2, n_3 are all non-positive. If any index is zero, the integral becomes trivial. Using the $\partial_1 \cdot v$ and $\partial_3 \cdot v$ relations

$$\left[-2n_1\mathbf{1}^+ + n_4\mathbf{4}^+(\mathbf{1}^- - 1) + n_6\mathbf{6}^+(\mathbf{1}^- - \mathbf{3}^-)\right] J_c = 0\,, \tag{3.22}$$

$$\left[-2n_3\mathbf{3}^+ + n_6\mathbf{6}^+(\mathbf{3}^- - \mathbf{1}^-) + n_7\mathbf{7}^+(\mathbf{3}^- - \mathbf{2}^-)\right] J_c = 0\,, \tag{3.23}$$

we can lower n_1, n_2 (symmetric case) and n_3 down to 1.

We are left with $J_c(1, 1, 1, n_4, n_5, n_6, n_7)$. It is rather difficult to apply the standard techniques to this integral, because each operator $\partial_i \cdot k_j$ produces a scalar product which cannot be expressed via the denominators. For example, $\partial_1 \cdot k_1$ gives the term $-\frac{n_6}{D_6} 2k_1 \cdot k_3$, and $\partial_1 \cdot (k_1 - k_3)$ gives the term $-\frac{n_4}{D_4} 2k_1 \cdot k_3$. We can cancel $2k_1 \cdot k_3$ by forming the difference

$$(n_4 - 1)\mathbf{6}^-\partial_1 \cdot k_1 - (n_6 - 1)\mathbf{4}^-\partial_1 \cdot (k_1 - k_3)\,,$$

which results in

$$\left[(n_4 - 1)\left(d - n_1 - 2n_4 + n_1\mathbf{1}^+\right)\mathbf{6}^- - (n_6 - 1)\left(d - n_1 - 2n_6 + n_1\mathbf{1}^+\mathbf{3}^-\right)\mathbf{4}^-\right] J_c = 0\,. \tag{3.24}$$

Some other useful combinations are

$$\begin{aligned}
&(n_6 - 1)\mathbf{7}^-\partial_3 \cdot (k_3 - k_1) - (n_7 - 1)\mathbf{6}^-\partial_3 \cdot (k_3 - k_2)\\
&(n_4 - 1)\mathbf{5}^-\left(\partial_1 + \partial_2 + \partial_3\right) \cdot k_1 - (n_5 - 1)\mathbf{4}^-\left(\partial_1 + \partial_2 + \partial_3\right) \cdot k_2\\
&(n_5 - 1)\mathbf{6}^-\partial_3 \cdot (k_3 - k_2) - (n_6 - 1)\mathbf{5}^-\left(\partial_1 + \partial_2 + \partial_3\right) \cdot k_1\\
&(n_4 - 1)\mathbf{6}^-\partial_3 \cdot (k_3 - k_2) - (n_6 - 1)\mathbf{4}^-\left(\partial_1 + \partial_2 + \partial_3\right) \cdot k_2\\
&(n_4 - 1)\mathbf{7}^-\left(\partial_1 + \partial_3\right) \cdot k_1 - (n_7 - 1)\mathbf{4}^-\left(\partial_1 + \partial_3\right) \cdot (k_3 - k_2)\,,
\end{aligned}$$

where all $\partial_1 + \partial_2 + \partial_3$ pieces can be simplified by the homogeneity relation. Using (3.22) and (3.23), we obtain at $n_1 = n_2 = n_3 = 1$

$$\begin{aligned}
&\Big[(n_4 - 1)\left[2(d - 2n_4 - 1) - n_4\mathbf{4}^+(1 - \mathbf{1}^-)\right]\mathbf{6}^- -\\
&- 2(n_6 - 1)(d - 2n_6 - 1 - \mathbf{1}^+\mathbf{3}^-)\mathbf{4}^- + (n_4 - 1)(n_6 - 1)(\mathbf{1}^- - \mathbf{3}^-)\Big] J_c = 0
\end{aligned} \tag{3.25}$$

$$[(n_6-1)(d-2n_6-1+\mathbf{3}^+\mathbf{1}^-)\mathbf{7}^- - (n_7-1)(d-2n_7-1+\mathbf{3}^+\mathbf{2}^-)\mathbf{6}^-]J_c = 0 \quad (3.26)$$

$$\Big[(n_4-1)\left[2(d-n_5-n_6-n_7)+(\mathbf{2}^+ + \mathbf{3}^+)\mathbf{1}^-\right]\mathbf{5}^- - \\ -(n_5-1)\left[2(d-n_4-n_6-n_7)+(\mathbf{1}^+ + \mathbf{3}^+)\mathbf{2}^-\right]\mathbf{4}^-\Big]J_c = 0 \quad (3.27)$$

$$\Big[(n_5-1)(d-2n_7-n_6+\mathbf{3}^+\mathbf{2}^-)\mathbf{6}^- + \\ +(n_6-1)\left[2(d-n_6-n_7)-n_5-1+(\mathbf{2}^+ + \mathbf{3}^+)\mathbf{1}^-\right]\mathbf{5}^- + \\ +(n_5-1)(n_6-1)(\mathbf{4}^- - \mathbf{7}^-)\Big]J_c = 0 \quad (3.28)$$

$$\Big[(n_4-1)(d-2n_7-n_6+\mathbf{3}^+\mathbf{2}^-)\mathbf{6}^- + \\ +(n_6-1)\left[2(d-n_6-n_7)-n_4-1+(\mathbf{4}^+ + \mathbf{3}^+)\mathbf{2}^-\right]\mathbf{4}^- + \\ +(n_4-1)(n_6-1)(\mathbf{5}^- - \mathbf{7}^-)\Big]J_c = 0 \quad (3.29)$$

$$\Big[(n_4-1)[2(d-2n_4-1-\mathbf{3}^+\mathbf{1}^-) - \\ -n_4\mathbf{4}^+(1-\mathbf{1}^-)+(n_7-1)\mathbf{6}^+(\mathbf{3}^- - \mathbf{2}^-)]\mathbf{7}^- - \\ -2(n_7-1)\left[d-2n_7-1+\mathbf{1}^+\mathbf{3}^- - (\mathbf{1}^+ + \mathbf{3}^+)\mathbf{2}^-\right]\mathbf{4}^-\Big]J_c = 0\,. \quad (3.30)$$

One more useful relation is obtained by adding (3.22), its mirror-symmetric, formula (3.23) and using the homogeneity relation:

$$\Big[2\left[3d-n_1-n_2-n_3-2(n_4+n_5+n_6+n_7)\right] - \\ -n_4\mathbf{4}^+(1-\mathbf{1}^-)-n_5\mathbf{5}^+(1-\mathbf{2}^-)\Big]J_c = 0\,. \quad (3.31)$$

When $n_4<0$ and $n_6\neq 1$, we can raise n_4 by (3.25); when $n_4<0$ and $n_5\neq 1$, we can raise n_4 by (3.27); when $n_4<0$ and $n_7\neq 1$, we can raise n_4 by (3.30); when $n_4<0$ and $n_5=n_6=n_7=1$, we can raise n_4 or n_5 by (3.31). The case $n_5<0$ is symmetric. When $n_6<0$ and $n_7\neq 1$, we can raise n_6 by (3.26); when $n_6<0$ and $n_4\neq 1$, we can raise n_6 by (3.29); when $n_6<0$ and $n_5\neq 1$, we can raise n_4 by (3.28); when $n_6<0$ and $n_4=n_5=n_7=1$, we can raise n_4 or n_5 by (3.31). The case $n_7<0$ is symmetric. When all the indices are positive, we can lower n_6 down to 1 (raising n_4) by (3.25) and lower n_7 down to 1 (raising n_5) by its mirror-symmetric. Finally, we can lower n_4 and n_5 down to 1 by (3.27) together with (3.31).

The integral $J_c(1,1,1,1,1,1,1)$ cannot be reduced to simpler ones (in contrast to the massless case [3]), and should be considered as a new basis integral. Its value is currently unknown, and its calculation is highly non-trivial.

3.3 Ladder with three heavy-quark lines

Now let's consider the diagram of figure 11*c*. We define

$$\int \frac{N^{n_0} d^d k_1 d^d k_2 d^d k_3}{D_1^{n_1} D_2^{n_2} D_3^{n_3} D_4^{n_4} D_5^{n_5} D_6^{n_6} D_7^{n_7} D_8^{n_8}} = \\ = -i\pi^{3d/2}(-2\omega)^{3d+2n_0-2\sum_{i=4}^{8} n_i} I_c(n_1, n_2, n_3, n_4, n_5, n_6, n_7, n_8; n_0)\,,$$
$$\begin{aligned} D_1 &= \frac{(k_1+p)\cdot v}{\omega}\,, & D_2 &= \frac{(k_2+p)\cdot v}{\omega}\,, & D_3 &= \frac{(k_3+p)\cdot v}{\omega}\,, \\ D_4 &= -k_1^2\,, & D_5 &= -k_2^2\,, & D_6 &= -(k_1-k_3)^2\,, \\ D_7 &= -(k_2-k_3)^2\,, & D_8 &= -k_3^2\,, & N &= 2k_1\cdot k_2\,. \end{aligned} \tag{3.32}$$

This integral is mirror-symmetric with respect to $1 \leftrightarrow 2$, $4 \leftrightarrow 5$, $6 \leftrightarrow 7$. It vanishes when the indices of the following groups of lines are non-positive: 14, 16, 46, 25, 27, 57, 132, 637, 368, 378, 687, 485, 487, 586.

First we are going to get rid of the numerator. When $n_7 \neq 1$, we can lower n_0 by the $\partial_3 \cdot (k_3 - k_1)$ relation

$$\Big[d - n_3 - n_7 - n_8 - 2n_6 + n_3\mathbf{3}^+\mathbf{1}^- + \\ + n_7\mathbf{7}^+(\mathbf{0}^+ + \mathbf{4}^- + \mathbf{5}^- - \mathbf{7}^-) + n_8\mathbf{8}^+(\mathbf{5}^- - \mathbf{7}^-)\Big] I_c = 0\,; \tag{3.33}$$

the case $n_6 \neq 0$ is symmetric. When $n_5 \neq 1$, we can lower n_0 by

$$\Big[2(d - n_5 - n_6 - n_7) + n_0 - n_2 - n_3 - n_8 + 2n_0\mathbf{0}^-\mathbf{4}^- + \\ + (n_2\mathbf{2}^+ + n_3\mathbf{3}^+)\mathbf{1}^- - n_5\mathbf{5}^+\mathbf{0}^+ + n_8\mathbf{8}^+(\mathbf{4}^- - \mathbf{6}^-)\Big] I_c = 0\,, \tag{3.34}$$

which is the $(\partial_1 + \partial_2 + \partial_3)\cdot k_1$ relation simplified by the homogeneity relation. the case $n_4 \neq 1$ is symmetric. When $n_1 \neq 1$, we can lower n_0 or raise n_4 or n_6 by the $\partial_1 \cdot v$ relation

$$\left[-2n_1\mathbf{1}^+ + n_0\mathbf{0}^-(\mathbf{2}^- - 1) + n_4\mathbf{4}^+(\mathbf{1}^- - 1) + n_6\mathbf{6}^+(\mathbf{1}^- - \mathbf{3}^-)\right] I_c = 0\,; \tag{3.35}$$

the case $n_2 \neq 1$ is symmetric. Finally, we can raise n_1 or n_6 by the $\partial_1 \cdot k_1$ relation

$$\left[d + n_0 - n_1 - n_6 - 2n_4 + n_1\mathbf{1}^+ + n_6\mathbf{6}^+(\mathbf{8}^- - \mathbf{4}^-)\right] I_c = 0\,. \tag{3.36}$$

Now we shall discuss the integral $I_c(n_1, n_2, n_3, n_4, n_5, n_6, n_7, n_8)$ without numerator ($n_0 = 0$). Applying $\partial_3 \cdot v$, $\partial_1 \cdot (k_1 - k_3)$ to the integrand of (3.32), we obtain the recurrence relations

$$\left[-2n_3\mathbf{3}^+ + n_6\mathbf{6}^+(\mathbf{1}^- - \mathbf{3}^-) + n_7\mathbf{7}^+(\mathbf{2}^- - \mathbf{3}^-) + n_8\mathbf{8}^+(1 - \mathbf{3}^-)\right] I_c = 0\,, \tag{3.37}$$
$$\left[d - n_1 - n_4 - 2n_6 + n_1\mathbf{1}^+\mathbf{3}^- + n_4\mathbf{4}^+(\mathbf{8}^- - \mathbf{6}^-)\right] I_c = 0\,. \tag{3.38}$$

Homogeneity in ω gives the relation identical with (3.20).

The cases $n_1 = 0$, $n_2 = 0$, $n_4 = 0$, $n_5 = 0$, $n_6 = 0$, $n_7 = 0$ are trivial. When $n_1 < 0$ and $n_6 \neq 1$, we can raise n_1 by (3.37); when $n_1 < 0$ and $n_4 \neq 1$, we can raise n_1 using the sum of (3.35) and (3.37); when $n_1 < 0$ and $n_4 = n_6 = 1$, we can raise n_1 or n_6 by (3.36). The case $n_2 < 0$ is symmetric. When $n_4 < 0$ and $n_6 \neq 1$, we can raise n_4 by (3.36); when $n_4 < 0$ and $n_1 \neq 1$, $n_6 = 1$, we can raise n_4 or n_6 by (3.35); when $n_4 < 0$ and $n_1 = n_6 = 1$, we can raise n_4 or n_1 by (3.38). The case $n_5 < 0$ is symmetric. When $n_6 < 0$ and $n_4 \neq 1$, we can raise n_6 by (3.38); when $n_6 < 0$ and $n_1 \neq 1$, $n_4 = 1$, we can raise n_6 or n_4 by (3.35); when $n_6 < 0$ and $n_1 = n_4 = 1$, we can raise n_6 or n_1 by (3.36). The case $n_7 < 0$ is symmetric. The case $n_3 = 0$ is $J_b(n_1, n_2, n_4, n_5, n_8, n_6, n_7)$ (figure 12a, section 3.1). When $n_3 < 0$ and $n_6 \neq 1$, we can raise n_3 by (3.35) (the case $n_7 \neq 1$ is symmetric); when $n_3 < 0$ and $n_8 \neq 1$, we can raise n_3 by the relation

$$\Big[2\left[3d - n_1 - n_2 - n_3 - 2(n_4 + n_5 + n_6 + n_7 + n_8)\right] + \\ + n_4 \mathbf{4}^+(\mathbf{1}^- - 1) + n_5 \mathbf{5}^+(\mathbf{2}^- - 1) + n_8 \mathbf{8}^+(\mathbf{3}^- - 1)\Big] I_c = 0\,, \qquad (3.39)$$

which is (3.35) plus its mirror-symmetric plus (3.37) simplified by the homogeneity relation; when $n_3 < 0$ and $n_1 \neq 1$, we can raise n_3 by (3.38) (the case $n_2 \neq 1$ is symmetric); when $n_3 < 0$ and $n_1 = n_2 = n_6 = n_8 = 1$, we can raise n_1, n_2 or n_3 by the homogeneity relation. The case $n_8 = 0$ is $J_c(n_1, n_2, n_3, n_4, n_5, n_6, n_7)$ (figure 12c, section 3.2). When $n_8 < 0$ and $n_6 \neq 1$, we can raise n_8 by (3.36) (the case $n_7 \neq 1$ is symmetric); when $n_8 < 0$ and $n_4 \neq 1$, we can raise n_8 by (3.38) (the case $n_5 \neq 1$ is symmetric); when $n_8 < 0$ and $n_3 \neq 1$, $n_4 = n_6 = n_7 = 1$, we can raise n_8, n_6 or n_7 by (3.37); when $n_8 < 0$ and $n_1 \neq 1$, $n_3 = n_4 = n_6 = n_7 = 1$, we can raise n_4 or n_6 by (3.35) (the case $n_2 \neq 1$ is symmetric); when $n_8 < 0$ and $n_1 = n_2 = n_3 = n_4 = n_5 = n_6 = n_7 = 1$, we can raise n_1, n_2 or n_3 the homogeneity relation. When all the indices are positive, we can kill one of the lines 3, 6, 8 by (3.38), or one of the lines 3, 7, 8 by its mirror-symmetric relation.

3.4 Non-planar diagram with three heavy-quark lines

Now let's consider the diagram of figure 11d. We define

$$\begin{aligned}
&\int \frac{N^{n_0} d^d k_1 d^d k_2 d^d k_3}{D_1^{n_1} D_2^{n_2} D_3^{n_3} D_4^{n_4} D_5^{n_5} D_6^{n_6} D_7^{n_7} D_8^{n_8}} = \\
&\quad = -i\pi^{3d/2} (-2\omega)^{3d+2n_0-2\sum_{i=4}^{8} n_i} I_d(n_1, n_2, n_3, n_4, n_5, n_6, n_7, n_8; n_0)\,, \\
&D_1 = \frac{(k_1+p)\cdot v}{\omega}\,, \qquad D_2 = \frac{(k_2+p)\cdot v}{\omega}\,, \qquad D_3 = \frac{(k_1+k_2-k_3+p)\cdot v}{\omega}\,, \\
&D_4 = -k_1^2\,, \qquad D_5 = -k_2^2\,, \qquad D_6 = -(k_1-k_3)^2\,, \\
&D_7 = -(k_2-k_3)^2\,, \qquad D_8 = -k_3^2\,, \qquad N = 2k_1\cdot k_2\,.
\end{aligned} \qquad (3.40)$$

This integral is mirror-symmetric with respect to $1 \leftrightarrow 2$, $4 \leftrightarrow 5$, $6 \leftrightarrow 7$. It vanishes when the indices of the following groups of lines are non-positive: 46, 57, 132, 485, 487, 586, 687, 314, 136, 325, 237, 148, 178, 258, 268, 417, 526, 637, 368, 378.

Applying $\partial_1 \cdot v$, $\partial_3 \cdot v$, $\partial_1 \cdot k_1$, $\partial_1 \cdot (k_1 - k_3)$, $(\partial_1 + \partial_3) \cdot k_3$, $\partial_3 \cdot (k_3 - k_1)$, $(\partial_1 + \partial_2 + \partial_3) \cdot k_1$ to the integrand of (3.40), we obtain the recurrence relations

$$\left[n_0 \mathbf{0}^- (\mathbf{2}^- - 1) - 2n_1 \mathbf{1}^+ - 2n_3 \mathbf{3}^+ + n_4 \mathbf{4}^+ (\mathbf{1}^- - 1) + n_6 \mathbf{6}^+ (\mathbf{3}^- - \mathbf{2}^-)\right] I_d = 0 \,, \quad (3.41)$$

$$\begin{aligned} \Big[& 2n_3 \mathbf{3}^+ + n_6 \mathbf{6}^+ (\mathbf{2}^- - \mathbf{3}^-) + n_7 \mathbf{7}^+ (\mathbf{1}^- - \mathbf{3}^-) + \\ & + n_8 \mathbf{8}^+ (\mathbf{1}^- + \mathbf{2}^- - \mathbf{3}^- - 1) \Big] I_d = 0 \,, \end{aligned} \quad (3.42)$$

$$\left[d + n_0 - n_1 - n_6 - 2n_4 + n_1 \mathbf{1}^+ + n_3 \mathbf{3}^+ (1 - \mathbf{1}^-) + n_6 \mathbf{6}^+ (\mathbf{8}^- - \mathbf{4}^-)\right] I_d = 0 \,, \quad (3.43)$$

$$\begin{aligned} \Big[& d + n_0 - n_3 - n_4 - 2n_6 + n_0 \mathbf{0}^- (\mathbf{5}^- - \mathbf{7}^- + \mathbf{8}^-) + \\ & + n_1 \mathbf{1}^+ (\mathbf{2}^- - \mathbf{3}^-) + n_3 \mathbf{3}^+ \mathbf{2}^- + n_4 \mathbf{4}^+ (\mathbf{8}^- - \mathbf{6}^-) \Big] I_d = 0 \,, \end{aligned} \quad (3.44)$$

$$\begin{aligned} \Big[& d - n_1 - n_4 - n_7 - 2n_8 + n_0 \mathbf{0}^- (\mathbf{7}^- - \mathbf{5}^- - \mathbf{8}^-) + \\ & + n_1 \mathbf{1}^+ (1 - \mathbf{2}^- + \mathbf{3}^-) + n_4 \mathbf{4}^+ (\mathbf{6}^- - \mathbf{8}^-) + n_7 \mathbf{7}^+ (\mathbf{5}^- - \mathbf{8}^-) \Big] I_d = 0 \,, \end{aligned} \quad (3.45)$$

$$\begin{aligned} \Big[& d - n_3 - n_7 - n_8 - 2n_6 + n_3 \mathbf{3}^+ \mathbf{2}^- + \\ & + n_7 \mathbf{7}^+ (\mathbf{0}^+ + \mathbf{4}^- + \mathbf{5}^- - \mathbf{6}^-) + n_8 \mathbf{8}^+ (\mathbf{4}^- - \mathbf{6}^-) \Big] I_d = 0 \,, \end{aligned} \quad (3.46)$$

$$\begin{aligned} \Big[& 2(d - n_5 - n_6 - n_7) + n_0 - n_2 - n_3 - n_8 + 2n_0 \mathbf{0}^- \mathbf{4}^- + \\ & + (n_2 \mathbf{2}^+ + n_3 \mathbf{3}^+) \mathbf{1}^- - n_5 \mathbf{5}^+ \mathbf{0}^+ + n_8 \mathbf{8}^+ (\mathbf{4}^- - \mathbf{6}^-) \Big] I_d = 0 \,, \end{aligned} \quad (3.47)$$

Homogeneity in ω gives

$$\begin{aligned} \Big[& 3d - n_1 - n_2 - n_3 - 2(n_4 + n_5 + n_6 + n_7 + n_8 - n_0) + \\ & + n_1 \mathbf{1}^+ + n_2 \mathbf{2}^+ + n_3 \mathbf{3}^+ \Big] I_d = 0 \end{aligned} \quad (3.48)$$

(this was used to simplify (3.47)).

The cases

$$\begin{aligned} I_d(0, n_2, n_3, n_4, n_5, n_6, n_7, n_8; n_0) &= (\mathbf{5}^- + \mathbf{6}^- - \mathbf{3}^- - \mathbf{8}^-)^{n_0} \times \\ & \times I_a(n_3, n_2, 0, n_5, n_6, n_7, n_4, n_8) \,, \end{aligned} \quad (3.49)$$

$$\begin{aligned} I_d(n_1, n_2, 0, n_4, n_5, n_6, n_7, n_8; n_0) &= (\mathbf{5}^- - \mathbf{3}^- - \mathbf{4}^-)^{n_0} \times \\ & \times I_a(n_1, n_2, n_4, n_5, 0, n_8, n_6, n_7) \end{aligned} \quad (3.50)$$

($n_2 = 0$ is symmetric to $n_1 = 0$) reduce to I_a (figure 11a, section 3.1). When $n_1 < 0$ and $n_4 \neq 1$, we can raise n_1 by (3.41); when $n_1 < 0$ and $n_7 \neq 1$, we can raise n_1 by the relation symmetric to (3.41); when $n_1 < 0$ and $n_8 \neq 1$, we can raise n_1 using the sum of (3.42) and the relation symmetric to (3.41); when $n_1 < 0$ and $n_3 \neq 1$, we can raise n_1 by (3.43); when $n_1 < 0$ and $n_2 \neq 1$, we can raise n_1 by the relation symmetric to (3.45); when $n_1 < 0$ and $n_2 = n_3 = n_4 = n_7 = n_8 = 1$, we can raise n_1, n_2 or n_3 by (3.48). The case $n_2 < 0$ is symmetric. When $n_3 < 0$ and $n_6 \neq 1$, we can raise n_3 by (3.41) (the case $n_7 \neq 1$ is symmetric); when $n_3 < 0$ and $n_8 \neq 1$, we can raise n_3 using the sum of (3.41), its symmetric and (3.42); when $n_3 < 0$ and $n_1 \neq 1$, we can raise n_3 by (3.44) (the case $n_2 \neq 1$ is symmetric); when $n_3 < 0$ and $n_1 = n_2 = n_6 = n_7 = n_8 = 1$, we can raise n_1, n_2 or n_3 by (3.48). When $n_1 > 1$, we can lower it using the sum of (3.41) and (3.42); the case $n_2 > 1$ is symmetric. When $n_3 > 1$, we can lower it by (3.42).

We are left with $I_d(1, 1, 1, n_4, n_5, n_6, n_7, n_8; n_0)$. Now we are going to get rid of the numerator. When $n_7 \neq 1$, we can lower n_0 by (3.46); the case $n_6 \neq 0$ is symmetric. When $n_5 \neq 1$, we can lower n_0 by (3.47); the case $n_4 \neq 1$ is symmetric. Otherwise, we use (3.44) to raise n_4, plus trivial terms.

Now we shall discuss the integral $I_d(1, 1, 1, n_4, n_5, n_6, n_7, n_8)$ without numerator ($n_0 = 0$). Using (3.41) and (3.42), we obtain from (3.48), (3.43) and (3.44) at $n_1 = n_2 = n_3 = 1$

$$\Big[2[3(d-1) - 2(n_4+n_5+n_6+n_7+n_8)] - n_4\mathbf{4}^+(1-\mathbf{1}^-) - \\ - n_5\mathbf{5}^+(1-\mathbf{2}^-) - n_8\mathbf{8}^+(1-\mathbf{1}^- - \mathbf{2}^- + \mathbf{3}^-)\Big] I_d = 0\,, \quad (3.51)$$

$$\Big[2(d - n_6 - 2n_4 - 1) - n_4\mathbf{4}^+(1-\mathbf{1}^-) + \\ + n_6\mathbf{6}^+(2(\mathbf{8}^- - \mathbf{4}^-) + \mathbf{3}^- - \mathbf{2}^-) - 2\mathbf{3}^+\mathbf{1}^-\Big] I_d = 0\,, \quad (3.52)$$

$$\Big[d - n_4 - 2n_6 - 1 + n_4\mathbf{4}^+(\mathbf{8}^- - \mathbf{6}^-) + \mathbf{1}^+(\mathbf{2}^- - \mathbf{3}^-)\Big] I_d = 0\,. \quad (3.53)$$

The case $n_8 = 0$ ($J_c(n_1, n_2, n_3, n_4, n_5, n_7, n_6)$, figure 12$c$, section 3.2) was discussed in section 3.2; the cases $n_6 = 0$ ($J_d(n_1, n_3, n_2, n_4, n_7, n_5, n_8)$, figure 12$d$; $n_7 = 0$ is symmetric) and $n_4 = 0$ ($J_e(n_1, n_3, n_2, n_6, n_8, n_5, n_7)$, figure 12$e$; $n_5 = 0$ is symmetric) will be discussed in section 3.5. If $n_4 < 0$, it can be raised by (3.53); the case $n_5 < 0$ is symmetric. When $n_6 < 0$ and $n_4 \neq 1$, we can raise n_6 by (3.53); when $n_6 < 0$ and $n_4 = 1$, we can raise n_6 or n_4 by (3.52). The case $n_7 < 0$ is symmetric. When $n_8 < 0$ and $n_4 \neq 1$, we can raise n_8 by (3.53) (the case $n_5 \neq 1$ is symmetric); when $n_8 < 0$ and $n_6 \neq 1$, we can raise n_8 by (3.52) (the case $n_7 \neq 1$ is symmetric); when $n_8 < 0$ and $n_4 = n_5 = n_6 = n_7 = 1$, we can raise n_8, n_4 or n_5 by (3.51). When the indices are positive, we can kill the line 6 or 8 using (3.53).

Thus, the non-planar diagram reduces to planar ones, in contrast to the massless case where such a reduction is impossible [3, 22].

3.5 Diagrams with four heavy-quark lines

Let's define (figure 11e)

$$
\begin{aligned}
&\int \frac{N_{13}^{n_{13}} N_{23}^{n_{23}} d^d k_1 d^d k_2 d^d k_3}{D_1^{n_1} D_2^{n_2} D_3^{n_3} D_4^{n_4} D_5^{n_5} D_6^{n_6} D_7^{n_7} D_8^{n_8}} = \\
&\qquad = -i\pi^{3d/2}(-2\omega)^{3d+2(n_{13}+n_{23})-2\sum_{i=5}^{8} n_i} I_e(n_1,n_2,n_3,n_4,n_5,n_6,n_7,n_8;n_{13},n_{23})\,, \\
&D_1 = \frac{(k_3+p)\cdot v}{\omega}\,, \qquad D_2 = \frac{(k_1+k_3+p)\cdot v}{\omega}\,, \qquad D_3 = \frac{(k_1+p)\cdot v}{\omega}\,, \\
&D_4 = \frac{(k_2+p)\cdot v}{\omega}\,, \qquad D_5 = -k_3^2\,, \qquad D_6 = -k_1^2\,, \\
&D_7 = -k_2^2\,, \qquad D_8 = -(k_1-k_2)^2\,, \\
&N_{13} = 2k_1\cdot k_3\,, \qquad N_{23} = 2k_2\cdot k_3\,.
\end{aligned} \tag{3.54}
$$

This integral vanishes when the indices of the following groups of lines are non-positive: 5, 12, 67, 68, 78, 47, 48, 234, 326, 238. The heavy-quark denominators are linearly dependent: $D_1 - D_2 + D_3 = 1$, and therefore

$$
\left[1 - \mathbf{1}^- + \mathbf{2}^- - \mathbf{3}^-\right] I_e = 0\,. \tag{3.55}
$$

If $n_1 < 0$, it can be raised by (3.55); if $n_2 < 0$, it can be raised by (3.55); if $n_3 < 0$, it can be raised by (3.55). If $n_{1,2,3}$ are all positive, we can lower them by (3.55) until one of them vanish. The cases

$$
\begin{aligned}
I_e(0,n_2,n_3,n_4,n_5,n_6,n_7,n_8;n_{13},n_{23}) &= (\mathbf{6}^- - \mathbf{4}^- + \mathbf{8}^-)^{n_{13}}(\mathbf{5}^- - \mathbf{7}^- + \mathbf{8}^- + \mathbf{0}^+)^{n_{23}} \times \\
&\quad \times I_c(n_2,n_4,n_3,0,n_7,n_5,n_8,n_6)\,, \\
I_e(n_1,0,n_3,n_4,n_5,n_6,n_7,n_8;n_{13},n_{23}) &= (\mathbf{6}^- - \mathbf{4}^- - \mathbf{8}^-)^{n_{13}}(\mathbf{0}^+)^{n_{23}} \times \\
&\quad \times I_c(n_1,n_4,n_3,n_5,n_7,0,n_8,n_6)\,, \\
I_e(n_1,n_2,0,n_4,n_5,n_6,n_7,n_8;n_{13},n_{23}) &= (\mathbf{4}^- - \mathbf{6}^- + \mathbf{8}^- + \mathbf{0}^+)^{n_{13}}(\mathbf{0}^+)^{n_{23}} \times \\
&\quad \times I_d(n_1,n_4,n_2,n_5,n_7,0,n_6,n_8)
\end{aligned}
$$

reduce to I_c (figure 11c, section 3.3) and I_d (figure 11d, section 3.4).

Let's define (figure 11f)

$$
\begin{aligned}
&\int \frac{N_{13}^{n_{13}} N_{23}^{n_{23}} d^d k_1 d^d k_2 d^d k_3}{D_1^{n_1} D_2^{n_2} D_3^{n_3} D_4^{n_4} D_5^{n_5} D_6^{n_6} D_7^{n_7} D_8^{n_8}} = \\
&\qquad = -i\pi^{3d/2}(-2\omega)^{3d+2(n_{13}+n_{23})-2\sum_{i=5}^{8} n_i} I_f(n_1,n_2,n_3,n_4,n_5,n_6,n_7,n_8;n_{13},n_{23})\,, \\
&D_1 = \frac{(k_1+p)\cdot v}{\omega}\,, \qquad D_2 = \frac{(k_1+k_3+p)\cdot v}{\omega}\,, \qquad D_3 = \frac{(k_2+k_3+p)\cdot v}{\omega}\,, \\
&D_4 = \frac{(k_2+p)\cdot v}{\omega}\,, \qquad D_5 = -k_3^2\,, \qquad D_6 = -k_1^2\,, \\
&D_7 = -k_2^2\,, \qquad D_8 = -(k_1-k_2)^2\,.
\end{aligned} \tag{3.56}
$$

This integral is mirror-symmetric with respect to $1 \leftrightarrow 4$, $2 \leftrightarrow 3$, $6 \leftrightarrow 7$. It vanishes when indices of the following groups of lines are non-positive: 5, 23, 67, 68, 78, 216, 128, 347, 438. The heavy-quark denominators are linearly dependent: $D_1 - D_2 + D_3 - D_4 = 0$, and therefore

$$\left[\mathbf{1}^- - \mathbf{2}^- + \mathbf{3}^- - \mathbf{4}^-\right] I_f = 0\,. \tag{3.57}$$

If $n_1 < 0$, it can be raised by (3.57) (the case $n_4 = 0$ is symmetric); if $n_2 < 0$, it can be raised by (3.57) (the case $n_3 = 0$ is symmetric). Otherwise, we can use (3.57) to raise, say, n_1 and kill one of the lines 2, 3, 4. The cases

$$\begin{aligned}
I_f(0, n_2, n_3, n_4, n_5, n_6, n_7, n_8; n_{13}, n_{23}) &= (\mathbf{7}^- - \mathbf{5}^- + \mathbf{8}^-)^{n_{13}} (\mathbf{8}^- - \mathbf{5}^- - \mathbf{7}^-)^{n_{23}} \times \\
&\quad \times I_d(n_2, n_4, n_3, 0, n_7, n_5, n_8, n_6)\,, \\
I_f(n_1, 0, n_3, n_4, n_5, n_6, n_7, n_8; n_{13}, n_{23}) &= (\mathbf{8}^- - \mathbf{4}^- - \mathbf{6}^-)^{n_{13}} (\mathbf{6}^- - \mathbf{4}^- + \mathbf{8}^-)^{n_{23}} \times \\
&\quad \times I_c(n_1, n_3, n_4, n_6, 0, n_8, n_5, n_7)
\end{aligned}$$

($n_3 = 0$ is symmetric to $n_2 = 0$, $n_4 = 0$ is symmetric to $n_1 = 0$) reduce to I_c (figure 11c, section 3.3) and I_d (figure 11d, section 3.4).

Let's define (figure 11g)

$$\begin{aligned}
&\int \frac{N_{13}^{n_{13}} N_{23}^{n_{23}} d^d k_1 d^d k_2 d^d k_3}{D_1^{n_1} D_2^{n_2} D_3^{n_3} D_4^{n_4} D_5^{n_5} D_6^{n_6} D_7^{n_7} D_8^{n_8}} = \\
&\quad = -i\pi^{3d/2} (-2\omega)^{3d + 2(n_{13}+n_{23}) - 2\sum_{i=4}^{8} n_i} I_g(n_1, n_2, n_3, n_4, n_5, n_6, n_7, n_8; n_{13}, n_{23}) \\
&D_1 = \frac{(k_3 + p)\cdot v}{\omega}\,, \qquad D_2 = \frac{(k_1 + k_3 + p)\cdot v}{\omega}\,, \qquad D_3 = \frac{(k_2 + k_3 + p)\cdot v}{\omega}\,, \\
&D_4 = \frac{(k_2 + p)\cdot v}{\omega}\,, \qquad D_5 = -k_3^2\,, \qquad D_6 = -k_1^2\,, \\
&D_7 = -k_2^2\,, \qquad D_8 = -(k_1 - k_2)^2\,.
\end{aligned} \tag{3.58}$$

This integral vanishes when the indices of the following groups of lines are non-positive: 5, 67, 68, 78, 26, 28, 123, 234, 347, 438. The heavy-quark denominators are linearly dependent: $D_1 - D_3 + D_4 = 1$, and therefore

$$\left[1 - \mathbf{1}^- + \mathbf{3}^- - \mathbf{4}^-\right] I_g = 0\,. \tag{3.59}$$

If $n_1 < 0$, it can be raised by (3.59); if $n_3 < 0$, it can be raised by (3.59); if $n_4 < 0$, it can be raised by (3.59). If $n_{1,3,4}$ are all positive, we can lower them by (3.59) until one of them vanish. The cases

$$\begin{aligned}
I_g(0, n_2, n_3, n_4, n_5, n_6, n_7, n_8; n_{13}, n_{23}) &= (\mathbf{6}^- - \mathbf{4}^- + \mathbf{8}^-)^{n_{13}} (\mathbf{5}^- - \mathbf{7}^- + \mathbf{8}^- + \mathbf{0}^+)^{n_{23}} \times \\
&\quad \times I_d(n_2, n_4, n_3, 0, n_7, n_5, n_8, n_6)\,, \\
I_g(n_1, n_2, 0, n_4, n_5, n_6, n_7, n_8; n_{13}, n_{23}) &= (\mathbf{4}^- - \mathbf{6}^- + \mathbf{8}^- + \mathbf{0}^+)^{n_{13}} (\mathbf{0}^+)^{n_{23}} \times \\
&\quad \times I_d(n_1, n_4, n_2, n_5, n_7, 0, n_6, n_8)\,, \\
I_g(n_1, n_2, n_3, 0, n_5, n_6, n_7, n_8; n_{13}, n_{23}) &= (\mathbf{4}^- + \mathbf{6}^- - \mathbf{0}^+)^{n_{13}} (\mathbf{4}^- - \mathbf{5}^- + \mathbf{8}^-)^{n_{23}} \times \\
&\quad \times I_b(n_1, n_3, n_2, n_5, 0, n_6, n_8, n_7)
\end{aligned}$$

reduce to I_b (figure 11b, section 3.2) and I_d (figure 11d, section 3.4).

Now we consider $J_d(n_1, n_2, n_3, n_5, n_6, n_7, n_8) = I_e(n_1, n_2, 0, n_3, n_5, n_6, n_7, n_8)$ (figure 12d), which is required for the calculation of the non-planar diagram I_d (figure 11d, section 3.4) without numerator. This integral vanishes when indices of the following groups of lines are non-positive: 5, 12, 23, 67, 68, 78, 26, 28, 37, 38. It becomes trivial if any of the indices is zero. Applying $\partial_3 \cdot v$, $\partial_1 \cdot v$, $\partial_2 \cdot v$, $\partial_3 \cdot k_3$, $\partial_1 \cdot k_1$, $\partial_1 \cdot (k_1 - k_2)$, $\partial_2 \cdot k_2$, $\partial_2 \cdot (k_2 - k_1)$ to the integrand, we obtain

$$\left[-2n_1\mathbf{1}^+ - 2n_2\mathbf{2}^+ + n_5\mathbf{5}^+(\mathbf{1}^- - 1)\right] J_d = 0 \tag{3.60}$$

$$\left[-2n_2\mathbf{2}^+ + n_6\mathbf{6}^+(\mathbf{2}^- - \mathbf{1}^-) + n_8\mathbf{8}^+(1 - \mathbf{1}^- + \mathbf{2}^- - \mathbf{3}^-)\right] J_d = 0 \tag{3.61}$$

$$\left[-2n_3\mathbf{3}^+ + n_7\mathbf{7}^+(\mathbf{3}^- - 1) - n_8\mathbf{8}^+(1 - \mathbf{1}^- + \mathbf{2}^- - \mathbf{3}^-)\right] J_d = 0 \tag{3.62}$$

$$\left[d - n_1 - 2n_5 + n_1\mathbf{1}^+ + n_2\mathbf{2}^+(1 - \mathbf{1}^-)\right] J_d = 0 \tag{3.63}$$

$$\left[d - n_1 - n_8 - 2n_6 + n_2\mathbf{2}^+\mathbf{1}^- + n_8\mathbf{8}^+(\mathbf{7}^- - \mathbf{6}^-)\right] J_d = 0 \tag{3.64}$$

$$\left[d - n_2 - n_6 - 2n_8 + n_2\mathbf{2}^+(\mathbf{1}^- + \mathbf{3}^- - 1) + n_6\mathbf{6}^+(\mathbf{7}^- - \mathbf{8}^-)\right] J_d = 0 \tag{3.65}$$

$$\left[d - n_3 - n_8 - 2n_7 + n_3\mathbf{3}^+ + n_8\mathbf{8}^+(\mathbf{6}^- - \mathbf{7}^-)\right] J_d = 0 \tag{3.66}$$

$$\left[d - n_3 - n_7 - 2n_8 + n_3\mathbf{3}^+(1 - \mathbf{1}^- + \mathbf{2}^-) + n_7\mathbf{7}^+(\mathbf{6}^- - \mathbf{8}^-)\right] J_d = 0\,. \tag{3.67}$$

Homogeneity in ω gives

$$\left[3d - n_1 - n_2 - n_3 - 2(n_5 + n_6 + n_7 + n_8) + n_1\mathbf{1}^+ + n_2\mathbf{2}^+ + n_3\mathbf{3}^+\right] J_d = 0\,. \tag{3.68}$$

When $n_1 < 0$ and $n_5 \neq 1$, we can raise n_1 by (3.60); when $n_1 < 0$ and $n_8 \neq 1$, we can raise n_1 by (3.62); when $n_1 < 0$ and $n_2 \neq 1$, we can raise n_1 by (3.64) (or (3.63)); when $n_1 < 0$ and $n_3 \neq 1$, we can raise n_1 by (3.67); when $n_1 < 0$ and $n_2 = n_3 = n_5 = n_8 = 1$, we can raise n_1, n_2 or n_3 by (3.68). When $n_2 < 0$ and $n_8 \neq 1$, we can raise n_2 by (3.62); when $n_2 < 0$ and $n_3 \neq 1$, we can raise n_2 by (3.67); when $n_2 < 0$ and $n_6 \neq 1$, we can raise n_2 using the sum of (3.61) and (3.62); when $n_2 < 0$ and $n_3 = n_6 = n_8 = 1$, we can raise n_2 or n_8 by (3.64). When $n_3 < 0$ and $n_8 \neq 1$, we can raise n_3 by (3.61); when $n_3 < 0$ and $n_7 \neq 1$, we can raise n_3 using the sum of (3.61) and (3.62); when $n_3 < 0$ and $n_2 \neq 1$, we can raise n_3 by (3.65); when $n_3 < 0$ and $n_2 = n_7 = n_8 = 1$, we can raise n_3 or n_8 by (3.66). When $n_6 < 0$ and $n_8 \neq 1$, we can raise n_6 by (3.64); when $n_6 < 0$ and $n_7 \neq 1$, we can raise n_6 by (3.67); when $n_6 < 0$ and $n_2 \neq 1$, $n_7 = n_8 = 1$, we can raise n_6 or n_8 by (3.61); when $n_6 < 0$ and $n_2 = n_7 = n_8 = 1$, we can raise n_6 or n_2 by (3.65). When $n_7 < 0$ and $n_8 \neq 1$, we can raise n_7 by (3.64); when $n_7 < 0$ and $n_6 \neq 1$, we can raise n_7 by (3.65); when $n_7 < 0$ and $n_2 \neq 1$, $n_6 = n_8 = 1$, we can raise n_6 or n_8 by (3.61); when $n_7 < 0$ and $n_2 = n_6 = n_8 = 1$, we can raise n_6 or n_2 by (3.65). When $n_8 < 0$ and $n_6 \neq 1$, we can raise n_8 by (3.65); when $n_8 < 0$ and $n_7 \neq 1$, we can raise n_8 by (3.67); when $n_8 < 0$ and $n_2 \neq 1$, $n_6 = n_7 = 1$, we can raise n_8 or n_6 by (3.61); when $n_8 < 0$ and $n_2 = n_6 = n_7 = 1$, we can raise n_6 or n_2 by (3.65). When all the indices are positive, we can kill one of the lines 1, 6, 7 by (3.64).

Finally, we consider $J_e(n_1, n_2, n_3, n_5, n_6, n_7, n_8) = I_f(0, n_1, n_2, n_3, n_5, n_6, n_7, n_8)$ (figure 12f), which is also required for the calculation of the non-planar diagram I_d (figure 11d, section 3.4) without numerator. This integral vanishes when indices of the following groups of lines are non-positive: 5, 12, 16, 18, 67, 68, 78, 237, 328. It becomes trivial if any of the indices is zero. Applying $\partial_3 \cdot v$, $\partial_1 \cdot v$, $\partial_2 \cdot v$, $\partial_3 \cdot k_3$, $\partial_1 \cdot k_1$, $\partial_1 \cdot (k_1 - k_2)$, $\partial_2 \cdot k_2$, $\partial_2 \cdot (k_2 - k_1)$ to the integrand, we obtain

$$\left[-2n_1\mathbf{1}^+ - 2n_2\mathbf{2}^+ + n_5\mathbf{5}^+(\mathbf{2}^- - \mathbf{3}^-)\right] J_e = 0 \quad (3.69)$$

$$\left[-2n_1\mathbf{1}^+ + n_6\mathbf{6}^+(\mathbf{1}^- - \mathbf{2}^- + \mathbf{3}^- - 1) + n_8\mathbf{8}^+(\mathbf{1}^- - \mathbf{2}^-)\right] J_e = 0 \quad (3.70)$$

$$\left[-2n_2\mathbf{2}^+ - 2n_3\mathbf{3}^+ + n_7\mathbf{7}^+(\mathbf{3}^- - 1) - n_8\mathbf{8}^+(\mathbf{2}^- - \mathbf{1}^-)\right] J_e = 0 \quad (3.71)$$

$$\left[d - n_2 - 2n_5 + n_1\mathbf{1}^+(\mathbf{3}^- - \mathbf{2}^-) + n_2\mathbf{2}^+\mathbf{3}^-\right] J_e = 0 \quad (3.72)$$

$$\left[d - n_1 - n_8 - 2n_6 + n_1\mathbf{1}^+(\mathbf{2}^- - \mathbf{3}^- + 1) + n_8\mathbf{8}^+(\mathbf{7}^- - \mathbf{6}^-)\right] J_e = 0 \quad (3.73)$$

$$\left[d - n_1 - n_6 - 2n_8 + n_1\mathbf{1}^+\mathbf{2}^- + n_6\mathbf{6}^+(\mathbf{7}^- - \mathbf{8}^-)\right] J_e = 0 \quad (3.74)$$

$$\left[d - n_3 - n_8 - 2n_7 + n_2\mathbf{2}^+(1 - \mathbf{3}^-) + n_3\mathbf{3}^+ + n_8\mathbf{8}^+(\mathbf{6}^- - \mathbf{7}^-)\right] J_e = 0 \quad (3.75)$$

$$\left[d - n_2 - n_7 - 2n_8 + n_2\mathbf{2}^+\mathbf{1}^- + n_3\mathbf{3}^+(\mathbf{1}^- - \mathbf{2}^-) + n_7\mathbf{7}^+(\mathbf{6}^- - \mathbf{8}^-)\right] J_e = 0\,. \quad (3.76)$$

Homogeneity in ω gives a relation identical to (3.68).

When $n_1 < 0$ and $n_8 \neq 1$, we can raise n_1 by (3.71); when $n_1 < 0$ and $n_6 \neq 1$, we can raise n_1 using the sum of (3.70) and (3.71); when $n_1 < 0$ and $n_6 = n_8 = 1$, we can raise n_1 or n_6 by (3.74). When $n_2 < 0$ and $n_5 \neq 1$, we can raise n_2 by (3.69); when $n_2 < 0$ and $n_8 \neq 1$, we can raise n_2 by (3.71); when $n_2 < 0$ and $n_6 \neq 1$, we can raise n_2 using the sum of (3.70) and (3.71); when $n_2 < 0$ and $n_1 \neq 1$, we can raise n_2 by (3.74); when $n_2 < 0$ and $n_3 \neq 1$, we can raise n_2 by (3.76); when $n_2 < 0$ and $n_1 = n_3 = n_5 = n_6 = n_8 = 1$, we can raise n_2, n_1 or n_3 by the homogeneity relation. When $n_3 < 0$ and $n_5 \neq 1$, we can raise n_3 by (3.69); when $n_3 < 0$ and $n_6 \neq 1$, we can raise n_3 by 93.70); when $n_3 < 0$ and $n_7 \neq 1$, we can raise n_3 by (3.71); when $n_3 < 0$ and $n_1 \neq 1$, we can raise n_3 by (3.73); when $n_3 < 0$ and $n_2 \neq 1$, we can raise n_3 using the sum of (3.72) and (3.73); when $n_3 < 0$ and $n_1 = n_2 = n_5 = n_6 = n_7 = 1$, we can raise n_3, n_1 or n_2 by the homogeneity relation. When $n_6 < 0$ and $n_8 \neq 1$, we can raise n_6 by (3.73); when $n_6 < 0$ and $n_7 \neq 1$, we can raise n_6 by (3.76); when $n_6 < 0$ and $n_2 \neq 1$, $n_7 = n_8 = 1$, we can raise n_6 or n_8 by (3.70); when $n_6 < 0$ and $n_2 = n_7 = n_8 = 1$, we can raise n_6 or n_2 by (3.74). When $n_7 < 0$ and $n_8 \neq 1$, we can raise n_7 by (3.73); when $n_7 < 0$ and $n_6 \neq 1$, we can raise n_7 by (3.74); when $n_7 < 0$ and $n_2 \neq 1$, $n_6 = n_8 = 1$, we can raise n_6 or n_8 by (3.70); when $n_7 < 0$ and $n_2 = n_6 = n_8 = 1$, we can raise n_6 or n_2 by (3.74). When $n_8 < 0$ and $n_6 \neq 1$, we can raise n_8 by (3.74); when $n_8 < 0$ and $n_7 \neq 1$, we can raise n_8 by (3.76); when $n_8 < 0$ and $n_2 \neq 1$, $n_6 = n_7 = 1$, we can raise n_8 or n_6 by (3.70); when $n_8 < 0$ and $n_2 = n_6 = n_7 = 1$, we can raise n_6 or n_2 by (3.74). When all the indices are positive, we can kill one of the lines 2, 7, 8 by (3.73).

3.6 Diagrams with five heavy-quark lines

Let's define (figure 11h)

$$\int \frac{N_{12}^{n_{12}} N_{13}^{n_{13}} N_{23}^{n_{23}} d^d k_1 d^d k_2 d^d k_3}{D_1^{n_1} D_2^{n_2} D_3^{n_3} D_4^{n_4} D_5^{n_5} D_6^{n_6} D_7^{n_7} D_8^{n_8}} = -i\pi^{3d/2} (-2\omega)^{3d+2(n_{12}+n_{13}+n_{23})-2\sum_{i=6}^{8} n_i} \times$$
$$\times I_h(n_1, n_2, n_3, n_4, n_5, n_6, n_7, n_8; n_{12}, n_{13}, n_{23})$$

$$\begin{aligned}
&D_1 = \frac{(k_2+p)\cdot v}{\omega}, && D_2 = \frac{(k_1+k_2+p)\cdot v}{\omega}, && D_3 = \frac{(k_1+p)\cdot v}{\omega},\\
&D_4 = \frac{(k_1+k_3+p)\cdot v}{\omega}, && D_5 = \frac{(k_3+p)\cdot v}{\omega}, && D_6 = -k_2^2,\\
&D_7 = -k_3^2, && D_8 = -k_1^2, &&\\
&N_{12} = 2k_1\cdot k_2, && N_{13} = 2k_1\cdot k_3, && N_{23} = 2k_2\cdot k_3.
\end{aligned} \tag{3.77}$$

This integral is mirror-symmetric with respect to $1 \leftrightarrow 5$, $2 \leftrightarrow 4$, $6 \leftrightarrow 7$. It vanishes when the indices of the following groups of lines are non-positive: 6, 7, 8, 12, 45, 234. There are two linear relations among the heavy-quark denominators:

$$\left[1 - \mathbf{1}^- + \mathbf{2}^- - \mathbf{3}^-\right] I_h = 0, \qquad \left[1 - \mathbf{3}^- + \mathbf{4}^- - \mathbf{5}^-\right] I_h = 0. \tag{3.78}$$

If $n_1 < 0$, $n_2 < 0$, $n_3 < 0$, $n_4 < 0$ or $n_5 < 0$, it can be raised by (3.78). Otherwise, one of the lines 1, 2, 3 (or 3, 4, 5) can be killed by (3.78). In the cases $n_1 = 0$ or $n_2 = 0$, we can use (3.78) to kill one of the lines 3, 4, 5; in the case $n_3 = 0$ (figure 12f), we can use

$$\left[\mathbf{1}^- - \mathbf{2}^- + \mathbf{3}^- - \mathbf{4}^-\right] I_h = 0 \tag{3.79}$$

to raise, say, n_4 and kill one of the lines 1, 2, 3. We are left with the integrals (figure 13)

$$\begin{aligned}
I_h(0, n_2, 0, n_4, n_5, n_6, n_7, n_8; n_{12}, n_{13}, n_{23}) &= K_c(n_2, n_5, n_4, n_6, n_7, n_8; n_{23}, n_{12}, n_{13})\\
I_h(0, n_2, n_3, 0, n_5, n_6, n_7, n_8; n_{12}, n_{13}, n_{23}) &= K_b(n_5, n_3, n_2, n_7, n_8, n_6; n_{13}, n_{23}, n_{12})\\
I_h(0, n_2, n_3, n_4, 0, n_6, n_7, n_8; n_{12}, n_{13}, n_{23}) &= K_c(n_2, n_4, n_3, n_6, n_7, n_8; n_{23}, n_{12}, n_{13})\\
I_h(n_1, 0, 0, n_4, n_5, n_6, n_7, n_8; n_{12}, n_{13}, n_{23}) &= K_b(n_1, n_5, n_4, n_6, n_7, n_8; n_{23}, n_{12}, n_{13})\\
I_h(n_1, 0, n_3, 0, n_5, n_6, n_7, n_8; n_{12}, n_{13}, n_{23}) &= K_a(n_1, n_5, n_3, n_6, n_7, n_8; n_{23}, n_{12}, n_{13}),
\end{aligned}$$

which will be discussed later in this section (the case $n_2 = n_5 = 0$ is symmetric to $n_1 = n_4 = 0$, and $n_3 = n_4 = 0$ — to $n_2 = n_3 = 0$).

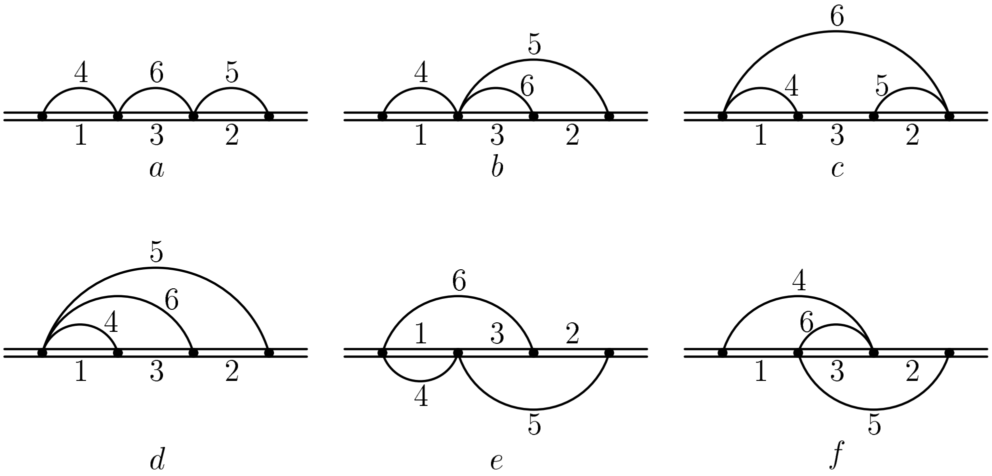


Figure 13: Reduced forms of the diagrams figure 11h–j

Let's define (figure 11i)

$$\begin{aligned}
&\int \frac{N_{12}^{n_{12}} N_{13}^{n_{13}} N_{23}^{n_{23}} d^d k_1 d^d k_2 d^d k_3}{D_1^{n_1} D_2^{n_2} D_3^{n_3} D_4^{n_4} D_5^{n_5} D_6^{n_6} D_7^{n_7} D_8^{n_8}} = -i\pi^{3d/2}(-2\omega)^{3d+2(n_{12}+n_{13}+n_{23})-2\sum_{i=6}^{8} n_i} \times \\
&\qquad\qquad \times I_i(n_1, n_2, n_3, n_4, n_5, n_6, n_7, n_8; n_{12}, n_{13}, n_{23})\,, \\
&D_1 = \frac{(k_1+p)\cdot v}{\omega}\,, \qquad D_2 = \frac{(k_1+k_2+p)\cdot v}{\omega}\,, \\
&D_3 = \frac{(k_1+k_2+k_3+p)\cdot v}{\omega}\,, \qquad D_4 = \frac{(k_1+k_3+p)\cdot v}{\omega}\,, \qquad D_5 = \frac{(k_2+p)\cdot v}{\omega}\,, \\
&D_6 = -k_1^2\,, \qquad D_7 = -k_2^2\,, \qquad D_8 = -k_3^2\,.
\end{aligned} \tag{3.80}$$

This integral vanishes when the indices of the following groups of lines are non-positive: 6, 7, 8, 34, 123. There are two linear relations among the heavy-quark denominators:

$$\left[1 - \mathbf{1}^- + \mathbf{3}^- - \mathbf{4}^-\right] I_i = 0\,, \qquad \left[1 - \mathbf{1}^- + \mathbf{2}^- - \mathbf{5}^-\right] I_i = 0\,. \tag{3.81}$$

If $n_1 < 0$, $n_2 < 0$, $n_3 < 0$, $n_4 < 0$, or $n_5 < 0$, it can be raised by (3.81). Otherwise, one of the lines 1, 3, 4 (or 1, 2, 5) can be killed by (3.81). In the cases $n_3 = 0$ or $n_4 = 0$, we can use (3.81) to kill one of the lines 1, 2, 5; in the case $n_1 = 0$, we can use

$$\left[\mathbf{2}^- - \mathbf{3}^- + \mathbf{4}^- - \mathbf{5}^-\right] I_i = 0 \tag{3.82}$$

to raise, say, n_5 and kill one of the lines 2, 3, 4. We are left with the integrals (figure 13)

$$\begin{aligned}
I_i(0,0,n_3,n_4,n_5,n_6,n_7,n_8;n_{12},n_{13},n_{23}) &= K_d(n_3,n_5,n_4,n_6,n_7,n_8;n_{12},n_{13},n_{23}) \\
I_i(0,n_2,0,n_4,n_5,n_6,n_7,n_8;n_{12},n_{13},n_{23}) &= K_c(n_2,n_4,n_5,n_6,n_8,n_7;n_{13},n_{12},n_{23}) \\
I_i(0,n_2,n_3,0,n_5,n_6,n_7,n_8;n_{12},n_{13},n_{23}) &= K_d(n_3,n_5,n_2,n_8,n_7,n_6;n_{23},n_{13},n_{12}) \\
I_i(n_1,0,0,n_4,n_5,n_6,n_7,n_8;n_{12},n_{13},n_{23}) &= K_b(n_1,n_5,n_4,n_6,n_7,n_8;n_{12},n_{13},n_{23})
\end{aligned}$$

$$
\begin{aligned}
I_i(n_1,0,n_3,0,n_5,n_6,n_7,n_8;n_{12},n_{13},n_{23}) &= K_f(n_1,n_5,n_3,n_6,n_7,n_8;n_{12},n_{13},n_{23})\\
I_i(n_1,n_2,0,n_4,0,n_6,n_7,n_8;n_{12},n_{13},n_{23}) &= K_e(n_4,n_1,n_2,n_8,n_6,n_7;n_{13},n_{23},n_{12})\\
I_i(n_1,n_2,n_3,0,0,n_6,n_7,n_8;n_{12},n_{13},n_{23}) &= K_d(n_3,n_1,n_2,n_8,n_6,n_7;n_{13},n_{23},n_{12})\,.
\end{aligned}
$$

The integrals K are shown in figure 13; the momenta of lines 4, 5, 6 are k_1, k_2, k_3. They are equal to

$$
\begin{aligned}
K_a(n_1,n_2,n_3,n_4,n_5,n_6;n_{12},n_{13},n_{23}) &= (\mathbf{0}^+)^{n_{12}}(\mathbf{6}^- - \mathbf{4}^- - \mathbf{8}^-)^{n_{13}}(\mathbf{7}^- - \mathbf{5}^- - \mathbf{8}^-)^{n_{23}}\times\\
&\quad \times I_c(n_1,n_2,n_3,n_4,n_5,0,0,n_6)\\
K_b(n_1,n_2,n_3,n_4,n_5,n_6;n_{12},n_{13},n_{23}) &= (\mathbf{0}^+)^{n_{12}}(\mathbf{6}^- - \mathbf{4}^- - \mathbf{8}^- - \mathbf{0}^+)^{n_{13}}\times\\
&\quad \times(\mathbf{7}^- - \mathbf{5}^- - \mathbf{8}^-)^{n_{23}}\times\\
&\quad \times I_c(n_1,n_2,n_3,n_4,n_5,0,n_6,0)\\
K_c(n_1,n_2,n_3,n_4,n_5,n_6;n_{12},n_{13},n_{23}) &= (\mathbf{4}^- + \mathbf{5}^- - \mathbf{6}^- - \mathbf{7}^- + \mathbf{0}^+)^{n_{12}}\times\\
&\quad \times(\mathbf{6}^- - \mathbf{4}^- + \mathbf{8}^-)^{n_{13}}(\mathbf{7}^- - \mathbf{5}^- + \mathbf{8}^-)^{n_{23}}\times\\
&\quad \times I_c(n_1,n_2,n_3,0,0,n_4,n_5,n_6)\\
K_d(n_1,n_2,n_3,n_4,n_5,n_6;n_{12},n_{13},n_{23}) &= (\mathbf{6}^- + \mathbf{7}^- - \mathbf{4}^- - \mathbf{5}^- - \mathbf{0}^+)^{n_{13}}\times\\
&\quad \times(\mathbf{5}^- - \mathbf{7}^- + \mathbf{8}^- + \mathbf{0}^+)^{n_{12}}(\mathbf{5}^- + \mathbf{7}^- - \mathbf{8}^-)^{n_{23}}\times\\
&\quad \times I_c(n_1,n_2,n_3,n_4,n_5,0,0,n_6)\\
K_e(n_1,n_2,n_3,n_4,n_5,n_6;n_{12},n_{13},n_{23}) &= (\mathbf{7}^- - \mathbf{5}^- + \mathbf{8}^-)^{n_{12}}(\mathbf{6}^- - \mathbf{4}^- + \mathbf{8}^-)^{n_{13}}\times\\
&\quad \times(\mathbf{5}^- - \mathbf{7}^- + \mathbf{8}^- + \mathbf{0}^+)^{n_{23}}\times\\
&\quad \times I_d(n_1,n_2,n_3,0,n_5,n_6,0,n_4)\\
K_f(n_1,n_2,n_3,n_4,n_5,n_6;n_{12},n_{13},n_{23}) &= (\mathbf{0}^+)^{n_{12}}(\mathbf{6}^- - \mathbf{4}^- - \mathbf{8}^-)^{n_{13}}(\mathbf{7}^- - \mathbf{5}^- - \mathbf{8}^-)^{n_{23}}\times\\
&\quad \times I_d(n_1,n_2,n_3,n_4,n_5,0,0,n_6)\,.
\end{aligned}
$$

We could also express K_b and K_d via I_b or I_d, and K_c via I_b.

Finally, let's define (figure 11j)

$$
\begin{aligned}
&\int \frac{N_{12}^{n_{12}} N_{13}^{n_{13}} N_{23}^{n_{23}} d^d k_1 d^d k_2 d^d k_3}{D_1^{n_1} D_2^{n_2} D_3^{n_3} D_4^{n_4} D_5^{n_5} D_6^{n_6} D_7^{n_7} D_8^{n_8}} = -i\pi^{3d/2}(-2\omega)^{3d+2(n_{12}+n_{13}+n_{23})-2\sum_{i=6}^{8} n_i}\times\\
&\qquad\qquad \times I_j(n_1,n_2,n_3,n_4,n_5,n_6,n_7,n_8;n_{12},n_{13},n_{23})\,,\\
&D_1 = \frac{(k_1+p)\cdot v}{\omega}\,, \qquad D_2 = \frac{(k_1+k_2+p)\cdot v}{\omega}\,,\\
&D_3 = \frac{(k_1+k_2+k_3+p)\cdot v}{\omega}\,, \qquad D_4 = \frac{(k_2+k_3+p)\cdot v}{\omega}\,, \qquad D_5 = \frac{(k_3+p)\cdot v}{\omega}\,,\\
&D_6 = -k_1^2\,, \qquad D_7 = -k_2^2\,, \qquad D_8 = -k_3^2\,.
\end{aligned}
\tag{3.83}
$$

This integral is mirror-symmetric with respect to $1 \leftrightarrow 5$, $2 \leftrightarrow 4$, $6 \leftrightarrow 8$. It vanishes when the indices of the following groups of lines are non-positive: 6, 7, 8, 123, 234, 345. The cases

$$
I_j(n_1,n_2,n_3,n_4,0,n_6,n_7,n_8) = I_i(n_1,n_2,n_3,n_4,0,n_6,n_7,n_8)
$$

$$I_j(n_1, 0, n_3, n_4, n_5, n_6, n_7, n_8) = I_i(n_1, 0, n_3, n_4, n_5, n_6, n_8, n_7)$$
$$I_j(n_1, n_2, 0, n_4, n_5, n_6, n_7, n_8) = I_h(n_1, n_2, 0, n_4, n_5, n_6, n_8, n_7)\,,$$

as well as the cases $n_1 = 0$, $n_4 = 0$, symmetric to the first two ones, reduce to the previously considered integrals. There are two linear relations among the heavy-quark denominators:

$$\left[1 - \mathbf{1}^- + \mathbf{3}^- - \mathbf{4}^-\right] I_j = 0\,, \qquad \left[1 - \mathbf{2}^- + \mathbf{3}^- - \mathbf{5}^-\right] I_j = 0\,. \tag{3.84}$$

If $n_1 < 0$, $n_2 < 0$, $n_3 < 0$, $n_4 < 0$, or $n_5 < 0$, it can be raised by (3.84). Otherwise, one of the lines 1, 3, 4 (or 2, 3, 5) can be killed by (3.84).

4. Implementation and testing

The package Grinder consists of a set of mutually recursive procedures for Feynman integrals of various topologies, which reduce a given Feynman integral to simpler ones, until boundary-case integrals with known values are reached. Remembering results of previous function calls may make the program run much faster, if there is enough memory. The package is written in REDUCE [23, 24]; (unfortunately, REDUCE uses linear look-up in remember-tables).

I also re-implemented it in Axiom [25]. All expressions involved are linear combinations of basis integrals with coefficients which are rational functions of d. It is convenient to use Axiom `Vector` domain, which has all the necessary operations. This makes intermediate expressions shorter than in the case when multivariate rational functions are used for entire expressions, because, typically, not all basis integrals are accompanied by every possible denominator. The amount of GCD calculations is thus reduced. This improvement can be, in principle, back-propagated to the REDUCE implementation by using matrices. However, working with matrices in REDUCE is awkward, because there are no local matrix variables, and no easy way for a function to return a matrix. I made no systematic benchmarking; one test set of 9375 recurrence relations took 2019 seconds in REDUCE and 350 seconds in Axiom. On the other hand, complete diagram calculations, including tensor and γ-matrix algebra, can be done in REDUCE.

The main method of testing was checking various recurrence relations (including those which were not directly used for construction of the algorithm) in nested loops over n_i. For each of two-loop and generalized two-loop integrals, which depend on 5 indices, a typical number of checks was about 20000; each test set of this size runs for a few hours. Integrals with 8 or 9 indices are more difficult to check. Some test sets required a few days of CPU time. A tool showing how many times each linear code segment has been executed would be invaluable for setting up test cases which check all branches at least once. Unfortunately, such a tool is not available in either programming system, and I had to emulate it by hand.

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